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Multivalued Variational Principles and Normed Coercivity [1]

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Abstract. A normed coercivity result is established for a class of order nonsmooth multivalued functionals fulfilling an appropriate Palais-Smale condition. The core of this approach is an asymptotic type statement involving such objects, obtained by means of a multivalued variational principle comparable with the one in Chen, Huang and Hou [J. Optimiz. Th. Appl., 106 (2000), 151-164].

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1 Introduction

Let $(X, \|,\|)$ be a (real) Banach space; and $(X^*, \|.\|)$, its *topological dual*. Given a (proper) functional $x \mapsto f(x)$ from X to R we say that it is *coercive*, if

(a01) $f(u) \to \infty$, provided $u \to \infty$ (in the sense: $\|u\| \to \infty$).

Sufficient conditions for deducing such a property involve a differential setting; and the most natural approach is a recursion to the celebrated 1964 Palais-Smale condition [23]. A typical result in this direction is the one due to Caklovic, Li and Willem [7]. Precisely, assume that

(a02) f is Gateaux differentiable and lower semicontinuous (lsc).

Then the property (a01) is deductible (when f is bounded from below) under a Palais-Smale requirement like

(a03) each sequence (v_n) in X with $(f(v_n))$ bounded and
$f'(v_n) \to 0$ (in X^*) has a convergent (in X) subsequence.

Note that (a02) holds when $f \in C^1(X)$; hence, their statement includes the one due to Brezis and Nirenberg [6]. An extension of this result (under the same condition (a02)) was obtained by Goeleven [14]. Specifically, the functional considered there is taken as $f = g+h$, where

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(a04) g is Gateaux differentiable lsc and h is (proper) convex lsc;

and the Palais-Smale condition (a03) is adapted to this decomposition. Some enlargements of this contribution were given in the paper by D. Motreanu and V. V. Motreanu [18]; where (a04) is to be taken as

(a05) g is locally Lipschitz and h is proper convex lsc

and the Palais-Smale requirement to be used is that in Motreanu and Panagiotopoulos [20, Ch 3]. Further aspects of "functional" nature were considered in Zhong [33] and Turinici [28].

A natural extension of all these facts is to be reached when (the univalued functional) $f : X \to R$ is being substituted by a multivalued mapping $x \mapsto F(x)$ from X to $\mathcal{P}_0(R) := \{Q \subseteq R; Q \neq \emptyset\}$. The basic result obtained in this direction is the one due to Kristaly and Varga [17]; and consists of two main ingredients:

i) a variational principle involving such maps, due to Chen, Huang and Hou [9], [10]

ii) a Palais-Smale condition based on contingent derivatives taken as in Aubin and Frankowska [3, Ch 4].

It is our aim in this exposition to show that further enlargements of this contribution are possible; details will be given in Section 4. The basic tool of it is the same variational principle in **i)**; which, as we shall see, may be viewed as a standard (univalued) one, with respect to the min-selection $f : X \to R$ (given as $f(x) = \min(F(x)); x \in X$); details will be given in Section 3. In addition, it is worth noting that, for deducing this variational statement, the Brezis-Browder principle [5] will suffice. And, the specific one is a multivalued version of the strong slope operator (for univalued functionals) introduced by DeGiorgi, Marino and Tosques [12]. Some particular versions of these developments are discussed in Section 5. Further aspects of the obtained facts will be delineated elsewhere.

2 Brezis-Browder statements

Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive transitive relation) $(\leq)$ over M; as well as a function $x \mapsto \varphi(x)$ from M to $R_+ := [0, \infty[$. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when: $w \in M$ and $z \leq w$ imply $\varphi(z) = \varphi(w)$. A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [5]:

Proposition 1 *Suppose that*

(b01) $(M, \leq)$ *is sequentially inductive:*
each ascending sequence has an upper bound (modulo $(\leq)$*)*

(b02) φ *is* $(\leq)$*-decreasing* $(x \leq y \Longrightarrow \varphi(x) \geq \varphi(y))$.

Then, for each $u \in M$ *there exists a* $(\leq, \varphi)$*-maximal* $v \in M$ *with* $u \leq v$.

Note that $\varphi(M) \subseteq R_+$ is not essential for the conclusion above; see Cârjă, Necula and Vrabie [8, Ch 2, Sect 2.1] for details. Moreover (as established there), Proposition 1 is reducible to the Principle of Dependent Choices (see, e.g., Wolk [31]). Finally (cf. Zhu and Li [34]), $(R_+, \geq)$ may be substituted by a separable ordered structure $(P, \leq)$ without altering the conclusion above; see also Turinici [29].

(A) This principle, including Ekeland's [13], found some useful applications to convex and nonconvex analysis (cf. the above references). For this reason, it was the subject of many extensions; such as the ones in Altman [1], Anisiu [2] and Szaz [25]. These are interesting from a technical viewpoint; but, whenever a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder's is always possible. This raises the question of to what extent are these enlargements of Proposition 1 effective. As we shall see, the answer is *essentially* negative; to do this, some conventions are needed. By a *pseudometric* over M we shall mean any map $d : M \times M \to R_+$. If, in addition, d is *reflexive* $[d(x, x) = 0, \forall x \in M]$, *triangular* $[d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in M]$ and *symmetric* $[d(x, y) = d(y, x), \forall x, y \in M]$, we say that it is a *semimetric* over M. Suppose that we fixed such an object. Call the point $z \in M$, *$(\leq, d)$-maximal*, in case: $w \in M$ and $z \leq w$ imply $d(z, w) = 0$. Note that, if (in addition) d is *sufficient* $[d(x, y) = 0$ implies $x = y]$, this property becomes: $w \in M, z \leq w \Longrightarrow z = w$ (and reads: z is *strongly* *$(\leq)$-maximal*). So, existence results involving such points may be viewed as "metrical" versions of the Zorn-Bourbaki maximality principle [35], [4]. To get sufficient conditions for these, one may proceed as below. Let (x_n) be an ascending sequence in M. The d-Cauchy property for it is introduced in the usual way: $\forall \varepsilon > 0, \exists n(\varepsilon)$ such that $n(\varepsilon) \leq p \leq q \Longrightarrow d(x_p, x_q) \leq \varepsilon$. Also, call this sequence *d-asymptotic*, when $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$. Clearly, each (ascending) d-Cauchy sequence is d-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent:

(b03) each ascending sequence is d-Cauchy

(b04) each ascending sequence is d-asymptotic.

By definition, either of these will be referred to as $(M, \leq)$ is *regular* (modulo d). Moreover, this property implies its relaxed version

(b05) $(M, \leq)$ is weakly regular (modulo d): $\forall x \in M, \forall \varepsilon > 0$,
$\exists y = y(x, \varepsilon) \geq x$ such that $y \leq u \leq v \Longrightarrow d(u, v) \leq \varepsilon$.

The following ordering principle is available (cf. Kang and Park [16]):

Proposition 2 *Assume that (b01) and (b05) are true. Then, for each $u \in M$ there exists a $(\leq, d)$-maximal $v \in M$ with $u \leq v$.*

As a direct consequence of this we have (cf. Turinici [26]):

Proposition 3 *Assume that $(M, \leq)$ is sequentially inductive and regular (modulo d). Then, the conclusion of Proposition 2 is retainable.*

Now (see the above reference) Prop 1 $\Longrightarrow$ Prop 2. On the other hand, Prop 2 $\Longrightarrow$ Prop 3 in a trivial way. Finally, Prop 3 $\Longrightarrow$ Prop 1; just take $d(x, y) = |\varphi(x) - \varphi(y)|$, $x, y \in M$ (where φ is the above one). Summing up, all these variants of the Brezis-Browder ordering principle (Proposition 1) are nothing but logical equivalents of it.

(B) A basic application of these facts is to "monotone" variational principles. Let M be a nonempty set. Take a quasi-order $(\leq)$ and a metric $d : M \times M \to R_+$ over it; the resulting triple will be termed a *quasi-ordered metric space*. Call the subset Z of M, $(\leq)$-*closed* when the limit of each ascending (modulo $(\leq)$) sequence in Z belongs to Z. Clearly, any closed part of M is $(\leq)$-closed too; but the converse is not in general true. (Just take $M = R$ (endowed with the usual order/metric); and $Z =]0, 1]$). Further, call the quasi-order $(\leq)$, *self-closed* provided $M(x, \leq) := \{u \in M; x \leq u\}$ is $(\leq)$-closed, for each $x \in M$; or, equivalently: the limit of each ascending sequence is an upper bound of it (modulo $(\leq)$). Finally, call the ambient metric d, $(\leq)$-*complete* provided each ascending (modulo $(\leq)$) d-Cauchy sequence converges. As before, if d is complete, then it is $(\leq)$-complete too. The reciprocal is not in general true; take $M =]0, 1]$ endowed with the standard order/metric.

We may now state the announced result. Take a function $\varphi : M \to R \cup \{\infty\}$ fulfilling

(b06) φ is inf-proper ($\mathrm{Dom}(\varphi) \neq \emptyset$ and $\varphi_* := \inf[\varphi(M)] > -\infty$)

(b07) φ is $(\leq)$-lsc over M: $[\varphi \leq t] := \{x \in X; \varphi(x) \leq t\}$ is $(\leq)$-closed, $\forall t \in R$.

Proposition 4 *Let $(\leq)$ be self-closed and d be $(\leq)$-complete. Then*
i) *for each $u \in \mathrm{Dom}(\varphi)$ there exists $v \in \mathrm{Dom}(\varphi)$ with*

$$u \leq v, d(u, v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v)\text{)} \tag{2.1}$$

$$d(v, x) > \varphi(v) - \varphi(x), \text{ for each } x \in M(v, \leq) \setminus \{v\}. \tag{2.2}$$

ii) *if $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ fulfill $\varphi(u) - \varphi_* \leq \rho$, then (2.1) gives*

$$(\varphi(u) \geq \varphi(v) \text{ and) } u \leq v, d(u, v) \leq \rho. \tag{2.3}$$

The original argument is that appearing in Turinici [27]. For the sake of completeness, we shall provide it, with some modifications.

Proof (of Proposition 4) Denote for simplicity $M[u] = \{x \in M; u \leq x, \varphi(u) \geq \varphi(x)\}$. Clearly, $\emptyset \neq M[u] \subseteq \mathrm{Dom}(\varphi)$; moreover, by (b07) (and the choice of $(\leq)$)

$$M[u] \text{ is } (\leq)\text{-closed; hence } d \text{ is } (\leq)\text{-complete on } M[u]. \tag{2.4}$$

Let $(\preceq)$ stand for the relation (over M): $x \preceq y$ iff $x \leq y, d(x, y) + \varphi(y) \leq \varphi(x)$. It is not hard to see that $(\preceq)$ acts as and *order* (antisymmetric quasi-order) on $\mathrm{Dom}(\varphi)$; so, it remains as such on $M[u]$. We claim that conditions of Proposition 3 are fulfilled on $(M[u]; \preceq; d)$. In fact, let (x_n) be an ascending (modulo $(\preceq)$) sequence in $M[u]$:

(b08) $x_n \leq x_m$ and $d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, if $n \leq m$.

The sequence $(\varphi(x_n))$ is descending and (by (b06)) bounded from below; hence a Cauchy one. This, along with the preceding relation, shows that (x_n) is an ascending (modulo $(\preceq)$) d-Cauchy sequence; wherefrom $(M[u], \preceq)$ is regular (modulo d). Moreover, the obtained properties give us (by (2.4)) some $y \in M[u]$ with $x_n \to y$. Combining with (b08) one derives (via (b07) and the choice of $(\leq)$)

$$x_n \leq y, d(x_n, y) \leq \varphi(x_n) - \varphi(y), \text{ (i.e.: } x_n \preceq y\text{), for all } n.$$

In other words, $y \in M[u]$ is an upper bound (modulo $(\preceq)$) of (x_n); and this shows that $(M[u], \preceq)$ is sequentially inductive. By Proposition 3 it then follows that, for the starting $u \in M[u]$ there exists $v \in M[u]$ with **j)** $u \preceq v$ and **jj)** v is $(\preceq, d)$-maximal in $M[u]$. The former of these is just (2.1). And the latter one gives at once (2.2); because it reads: $x \in M[u]$ and $v \preceq x$ imply $v = x$. The last part is evident; so, the conclusion follows. □

A basic particular case of our developments corresponds to the choice $(\leq) = M \times M$ (=the trivial quasi-order on M). The regularity condition (b07) may then be written as

(b09) φ is lsc over M: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \to x$;

and Proposition 4 is nothing but Ekeland's variational principle [13] (in short: EVP). On the other hand, the same requirement holds under (b02) and the self-closeness of $(\leq)$. For this reason, Proposition 4 will be called the *monotone* version of EVP. Note that, by the remarks above, it may be derived from Proposition 1 as well; we do not give details. Further aspects may be found in Hyers, Isac and Rassias [15, Ch 5].

3 Multivalued variational principles

Let (X, d) be a complete metric space; and $F : X \to \mathcal{P}_0(R)$ be a multivalued map from X to R. As usually, we shall identify F with its graph in $X \times R$; so, F is a relation between X and R with $\mathrm{Dom}(F) = X$. Define a quasi-order $(\preceq)$ over $X \times R$ as

(c01) $(x, t) \preceq (y, s)$ iff $d(x, y) \leq t - s$.

Clearly, it is in addition antisymmetric; hence a (partial) order. The question to be posed is the following: under which conditions about our data is $(\preceq)$, *admissible* over F in the Zorn-Bourbaki sense [each point of F is majorized by a $(\preceq)$-maximal one]. For an appropriate answer, note that, in the univalued case, these are **(c)** the boundedness from below and **cc)** the lsc property (see, for instance, Ekeland [13]). It is our aim to show that this is formally retainable in our multivalued case too. Precisely, assume that

(c02) F is bounded from below: $F_* := \inf[F(X)] > -\infty$

(c03) F is submonotone: for each $((x_n, t_n)) \subseteq F$ with $x_n \to x$ and (t_n) descending,
there exists $t \in F(x)$ with $t_n \geq t$, for all n.

Theorem 1 *Let these conditions hold. Then, for each* $(x_0, t_0) \in F$ *there exists* $(\bar{x}, \bar{t}) \in F$ *with* **i)** $(x_0, t_0) \preceq (\bar{x}, \bar{t})$*,* **ii)** $(\bar{x}, \bar{t}) \preceq (x, t) \in F \Longrightarrow (\bar{x}, \bar{t}) = (x, t)$ *and* **iii)** $\bar{t} = \inf[F(\bar{x})]$.

Proof Let e stand for the "product" metric: $e((x_1, t_1), (x_2, t_2)) = d(x_1, x_2) + |t_1 - t_2|$, $(x_1, t_1), (x_2, t_2) \in X \times R$. We claim that Proposition 3 is applicable to $(F; \preceq; e)$; and this will complete the argument. Let $((x_n, t_n)) \subseteq F$ be a $(\preceq)$-ascending sequence; i.e.,

(c04) $d(x_n, x_m) \leq t_n - t_m$, whenever $n \leq m$.

The sequence (t_n) is descending and bounded from below in $F(X)$ (by (c02)); hence, a Cauchy one. This, along with (c04), shows that (x_n) is d-Cauchy in $X = \mathrm{Dom}(F)$; and so, $(F, \preceq)$ is regular (modulo e). As (X, d) is complete, $x_n \to x$ for some $x \in X$; so (combining with (c03) above) there must be a $t \in F(x)$ with $t_n \geq t$, $\forall n$. By (c04),

$$(\forall n): \quad d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) \leq t_n - t + d(x_m, x), \forall m \geq n.$$

Passing to limit as $m \to \infty$ gives $d(x_n, x) \leq t_n - t$ (i.e.: $(x_n, t_n) \preceq (x, t)$), $\forall n$; which tells us that $(x, t) \in F$ is an upper bound of $((x_n, t_n))$; i.e., $(F, \preceq)$ is sequentially inductive; hence the claim. By Proposition 3 it then follows that, for the starting $(x_0, t_0) \in F$ there exists a $(\bar{x}, \bar{t}) \in F$ with the properties **i)** and **ii)** in the statement. Finally, note that (by (c02)) $-\infty < \inf[F(\bar{x})] \leq \bar{t}$. Assume by absurd that $\bar{t} > \inf[F(\bar{x})]$. By definition, there must be some $t^* \in F(\bar{x})$ with $\bar{t} > t^*$. But then, $(\bar{x}, \bar{t}) \preceq (\bar{x}, t^*)$, $(\bar{x}, \bar{t}) \neq (\bar{x}, t^*)$; in contradiction with the maximality of $(\bar{x}, \bar{t})$ in $(F, \preceq)$. Hence, **iii)** holds as well; and conclusion follows. □

Note that (c03) holds (under (c02)) when

(c05) F is closed: $((x_n, t_n)) \subseteq F$, $x_n \to x$, $t_n \to t$ imply $(x, t) \in F$.

For, if $((x_n, t_n))$ is as in the premise of (c03), $t = \lim_n(t_n)$ exists; and then $t \in F(x)$ by (c05). In this case, the corresponding version of Theorem 1 is compatible with the related statement in Phelps [24, Ch 3, Sect 3.12]. So, a discussion of (c05) would be useful. This will necessitate some conventions. We say thay F is *upper semicontinuous* at $x \in X$ if

(c06) $\forall \varepsilon > 0$, $\exists \delta > 0$: $x \in X(x, \delta) \Longrightarrow F(x) \subseteq R(F(x), \varepsilon)$.

Here, for each metric space (Z, e) and each couple $(W \subseteq Z; \gamma > 0)$ we put $Z(W, \gamma) = \{z \in Z; \mathrm{dist}(z, W) < \gamma\}$; where "dist" is the associated point to set distance in Z. We now claim that (c05) is obtainable from

(c07) F is (nonempty) closed valued: $(\emptyset \neq)$ $F(x) = $ closed, $\forall x \in X$

(c08) F is upper semicontinuous on X (i.e.: (c06) holds, for each $x \in X$).

In fact, let $(x_n, t_n)) \subseteq F$ be as in the premise of this condition. Given the arbitrary fixed $\varepsilon > 0$, let $\delta > 0$ be the number appearing in (c08). As $x_n \to x$, there must be some $n(\delta)$ in such a way that $n \geq n(\delta)$ implies $d(x_n, x) < \delta$. This, along with (c06), yields (for all such n) $F(x_n) \subseteq R(F(x), \varepsilon)$; hence $t_n \in R(F(x), \varepsilon)$, $\forall n \geq n(\delta)$. Passing to limit as $n \to \infty$ gives $\mathrm{dist}(t, F(x)) \leq \varepsilon$; so that (by the arbitrariness of $\varepsilon > 0$ and (c07)) $t \in F(x)$; hence the claim.

(B) An interesting particular case of these developments corresponds to F being univalued. So, let $f : X \to R$ be some function with

(c09) f is bounded from below ($f_* := \inf[f(X)] > -\infty$)

(c10) f is submonotone: for each $(x_n) \subseteq X$ with $x_n \to x$ and $(f(x_n))$ descending, we have $f(x_n) \geq f(x)$, $\forall n$.

Theorem 2 *Let these conditions hold. Then, for each $x_0 \in X$ there exists $\bar{x} \in X$ with* **j)** $d(x_0, \bar{x}) \leq f(x_0) - f(\bar{x})$, **jj)** $x \in X, d(\bar{x}, x) \leq f(\bar{x}) - f(x) \Longrightarrow \bar{x} = x$.

Note that (c10) is just the descending-lsc property used by Nemeth [22]. In fact, Theorem 2 gives an extended version of Ekeland's variational principle [13] (in short: EVP) based on (c09)+(c10). For, let $f : X \to R \cup \{\infty\}$ with $\mathrm{Dom}(f) \neq \emptyset$ be taken as precise; and $x_0 \in \mathrm{Dom}(f)$ be arbitrary fixed. Then, an application of Theorem 2 to $X_0 := \{x \in X; f(x) \leq f(x_0)\}$ gives the conclusions **j)** and **jj)** above; hence the claim. Clearly, this is also true for the standard form of EVP, with (c10) substituted by its stronger counterpart (b09) (modulo f). A natural question to be posed is that of the reciprocal implication (c10) $\Longrightarrow$ (b09) being available. The answer to this is negative; as shown by

Example 1 Take $X = R$ (endowed with the usual order/metric); and define

$$(f : R \to R) \quad f(t) = e^t - 1, \text{ if } t < 0;\ f(t) = t + 2, \text{ if } t \geq 0.$$

Clearly, (b09) does not hold for this object; because $\{t \in R; f(t) \leq 1\} =]-\infty, 0[$ is not closed. On the other hand, (c10) is retainable for f. In fact, let (t_n) in R and $t \in R$ be such that $t_n \to t$ and $(f(t_n))$ is descending. As f is strictly increasing, (t_n) is descending too. As a consequence, $t_n \geq t$, for all n; wherefrom $f(t_n) \geq f(t)$, for all n; hence the claim.

(C) The above developments raise the question of to what extent is Theorem 1 deductible from Theorem 2. Let $F : X \to \mathcal{P}_0(R)$ be such that (c02)+(c03) hold. Call the subset $P \subseteq R$, *inf-closed* provided $-\infty < \inf(P) \in P$.

Lemma 1 *Under these conditions, we have*

$$F(x) \text{ is inf-closed, for each } x \in X. \tag{3.1}$$

Proof By (c02), we have for the moment $\inf[F(x)] > -\infty$. Let $(t_n) \subseteq F(x)$ be descending and $\lim_n(t_n) = \inf[F(x)]$. By (c03) applied to the sequence $((x, t_n)) \subseteq F$ there exists $t \in F(x)$ with $t_n \geq t$, for all n. But then, $t = \inf[F(x)]$; hence the conclusion. □

As a consequence, the function $f : X \to R$ given by

(c11) $f(x) = \inf[F(x)]$, $x \in X$ (in short: $f = \inf[F]$)

is well defined. Moreover, by (3.1), f is a selection of F [i.e.: $f(x) \in F(x)$, $\forall x \in X$] with: $f \leq g$, for any other selection g of F. The question to be posed is: which properties of F are transferable to f. A partial answer to this is contained in

Lemma 2 *Let the precise conditions (c02)+(c03) hold. Then, $f := \inf[F]$ is bounded below and submonotone (in the sense of (c09)+(c10)).*

Proof Let $(x_n) \subseteq X$ be such that $x_n \to x$ and $(f(x_n))$ is descending. By (c03), there must be $t \in F(x)$ such that $f(x_n) \geq t$, for all n. On the other hand (by definition) $t \geq f(x)$. Combining these gives the conclusion. □

We are now in position to give an appropriate answer to the question above.

Proposition 5 *Under the precise conditions, we have Theorem 2 $\Longrightarrow$ Theorem 1; hence Theorem 2 $\Longleftrightarrow$ Theorem 1.*

Proof Let the premises of Theorem 1 be accepted; and $(x_0, t_0) \in F$ be arbitrary fixed. By Lemma 2, conditions (c09)+(c10) are valid for $f := \inf[F]$. Hence, Theorem 2 applies here; so that (by the notations we already introduced) for the starting $(x_0, f(x_0)) \in F$ there exists $(\bar{x}, f(\bar{x})) \in F$ with **h)** $(x_0, t_0) \preceq (\bar{x}, f(\bar{x}))$ and **hh)** $(\bar{x}, f(\bar{x})) \preceq (x, f(x)) \Longrightarrow \bar{x} = x$. Combining with $(x_0, t_0) \preceq (x_0, f(x_0))$ yields **i)** with $\bar{t} = f(\bar{x})$. Moreover, **iii)** holds as well with such a choice; so, it remains to verify **ii)**. Let $(x, t) \in F$ be such that $(\bar{x}, f(\bar{x})) \preceq (x, t)$. Since $(x, t) \preceq (x, f(x))$, we get $(\bar{x}, f(\bar{x})) \preceq (x, f(x))$; wherefrom (by **hh)** above) $\bar{x} = x$ (hence $f(\bar{x}) = f(x)$). This finally combined with $f(\bar{x}) \geq t \geq f(x)$ yields $f(\bar{x}) = t$; hence $(\bar{x}, f(\bar{x})) = (x, t)$; and the conclusion follows. □

Summing up, the multivalued versions of such maximal principles are non-effective. So, genuine extensions are possible in a vectorial context; we shall discuss them elsewhere.

4 Normed coercivity

With this information at hand, we may now return to the questions of the introductory part. Let $(X, \|,\|)$ be a (real) Banach space. Denote by d the (standard) metric induced by $||.||$; hence, it is invariant to translations and (X, d) is complete. Further, take some map $\Gamma : X \to R_+$ with the properties

(d01) Γ is almost (λ, μ)-Lipschitz ($d(x, y) \leq \lambda \Longrightarrow |\Gamma(x) - \Gamma(y)| \leq \mu$) for certain $\lambda, \mu > 0$ with $\lambda < 1 < \mu$

(d02) $\Gamma(X)$ has arbitrarily large points: $\sup[\Gamma(X)] = \infty$.

A useful consequence involving the level sets $\{[\Gamma \geq \sigma] := \{x \in X; \Gamma(x) \geq \sigma\}, \sigma > 0\}$ is

$$\mathrm{cl}([\Gamma \geq \rho]) \subseteq X([\Gamma \geq \rho], \lambda) \subseteq [\Gamma \geq \rho - \mu],\ \forall \rho \geq \mu; \tag{4.1}$$

where, "cl" is the closure operator. In fact, let $v \in X([\Gamma \geq \rho], \lambda)$ be arbitrary fixed. By definition, there must be some $u \in [\Gamma \geq \rho]$ with $d(u, v) < \lambda$; hence $|\Gamma(u) - \Gamma(v)| \leq \mu$ (if we take (d01) into account). But then, $\Gamma(v) \geq \Gamma(u) - \mu \geq \rho - \mu$ (i.e.: $v \in [\Gamma \geq \rho - \mu]$); and the claim follows. [Notice that all these level sets are nonvoid, by (d02); wherefrom, the reasoning above is effective]. Finally, pick some multivalued functional $F : X \to \mathcal{P}_0(R)$ with

(d03) F is bounded below and submonotone (cf. (c02)+(c03)).

Remember that, as a consequence of this, the (univalued) functional $f = \inf[F]$ (from X to R) is well defined (via (c11)) as a selection of F (minimal with respect to this property). In addition (see above)

$$f \text{ is bounded below and submonotone (in the sense of (c09)+(c10)).} \tag{4.2}$$

By these remarks, the quantities $m(\Gamma, f)(\sigma) := \inf[f([\Gamma \geq \sigma])]$ exist in R for each $\sigma > 0$. Moreover, the map $\sigma \mapsto m(\Gamma, f)(\sigma)$ is increasing from $R^0_+ :=]0, \infty[$ to R; wherefrom

$$\liminf_{\Gamma(u)\to\infty} f(u) := \sup_{\sigma>0} m(\Gamma, f)(\sigma)\ [= \lim_{\sigma\to\infty} m(\Gamma, f)(\sigma)]$$

exists, as an element of $R \cup \{\infty\}$, in view of

$$f_* \leq m(\Gamma, f)(\sigma) \leq \alpha(\Gamma, f) := \liminf_{\Gamma(u)\to\infty} f(u) \leq \infty, \quad \forall \sigma > 0. \tag{4.3}$$

When $\alpha(\Gamma, f) = \infty$, the functional f will be referred to as Γ-*coercive*; and the same convention applies to F. It is our aim in the following to get sufficient conditions in order that such a property be attained. These, as a rule, require a *differential* setting relative to f and F. Denote, for each $u \in X$,

(d04) $|\nabla| f(u) = \max\{0, \nabla f(u) := \limsup_{x\to u} \dfrac{f(u) - f(x)}{d(u,x)}\}.$

This object is comparable with the one introduced by DeGiorgi, Marino and Tosques [12]; and will be referred to as the strong *d-slope* of f at u. The usefulness of such concepts for the critical point theory (for univalued functionals) was underlined by Corvellec, DeGiovanni and Marzocchi [11]. Here, we shall establish that a certain multivalued version of it is the natural tool for our "multivalued" coercivity theory as well. Denote, for $(u, v) \in F$,

(d05) $|\nabla| F(u, v) = \max\{0, \nabla F(u) := \limsup_{x\to u} \dfrac{v - F(x)}{d(u,x)}\}.$

As before, we shall term this quantity, the strong *d-slope* of F at (u, v). The connection between these two concepts is discussed in the lemma below. For the subset P of R and the point $r \in R$, put $P \geq r$ whenever $t \geq r$, for each $t \in P$.

Lemma 3 *Under the precise conditions we have, for each* $u \in X$,

$$\nabla F(u, f(u)) \leq \nabla f(u);\ \text{hence } |\nabla| F(u, f(u)) \leq |\nabla| f(u). \tag{4.4}$$

Proof Let $\varepsilon > 0$ be arbitrary fixed. By the very definition of f,

$$\sup_{d(u,x)<\varepsilon} \frac{f(u) - F(x)}{d(u,x)} \leq \sup_{d(u,x)<\varepsilon} \frac{f(u) - f(x)}{d(u,x)}; \quad \text{as } F(x) \geq f(x), \text{ for all such } x.$$

Passing to infimum (=limit) as $\varepsilon \to 0$ yields the needed conclusion. □

The following asymptotic type statement is a basic step to the answer we are looking for.

Theorem 3 *Suppose that*

(d06) $\alpha(\Gamma, f) < \infty$ *(hence (cf. (4.3))* $\alpha(\Gamma, f)$ *is finite).*

There exists then, a sequence (v_n) *in* $\Gamma^{-1}(R^0_+)$ *with*

$$\Gamma(v_n) \to \infty \ (\text{so, } \Gamma(y_n) \to \infty \text{ for each subsequence } (y_n) \text{ of } (v_n)) \tag{4.5}$$

$$f(v_n) \to \alpha(\Gamma, F) \quad \text{as } n \to \infty \tag{4.6}$$

$$|\nabla| f(v_n) \to 0 \ (\text{hence } |\nabla| F(v_n, f(v_n)) \to 0). \tag{4.7}$$

Proof There are two steps to be passed.

(I) Let the parameter $\eta > 0$ be taken according to

(d07) $\eta < \frac{\lambda}{2\mu}$; hence (according to (d01)) $\frac{1}{\eta} > \mu > \frac{\lambda}{2} > \eta$.

By (d06), there exists $r(\eta)$ with

$$r(\eta) \geq 1/\eta; \quad m(\Gamma, f)(r) > \alpha(\Gamma, f) - \eta^2, \forall r \geq r(\eta). \tag{4.8}$$

Having this precise, we claim that there exists $v_\eta \in X$ so that

$$\Gamma(v_\eta) \geq r(\eta), |f(v_\eta) - \alpha(\Gamma, f)| < \eta^2, \tag{4.9}$$

$$|\nabla|F(v_\eta, f(v_\eta)) \leq |\nabla|f(v_\eta) \leq \eta. \tag{4.10}$$

In fact, by (4.8) (and the definition of these quantities) we have an evaluation like $\alpha(\Gamma, f) - \eta^2 < m(\Gamma, f)(4r(\eta)) < \alpha(\Gamma, f) + \eta^2$; wherefrom

$$f(u_\eta) < \alpha(\Gamma, f) + \eta^2, \text{ for some } u_\eta \in [\Gamma \geq 4r(\eta)].$$

Taking (d03) into account, it results that Theorem 2 applies to [$M = \mathrm{cl}[\Gamma \geq 2r(\eta)]$; d=as before; $g = (1/\eta)f$]. So, for the starting $u_\eta \in M$ there must be some $v_\eta \in M$ with

$$\eta d(u_\eta, v_\eta) \leq f(u_\eta) - f(v_\eta) \text{ (hence } f(u_\eta) \geq f(v_\eta)) \tag{4.11}$$

$$\eta d(v_\eta, x) > f(v_\eta) - f(x), \text{ for all } x \in M \setminus \{v_\eta\}. \tag{4.12}$$

We claim that v_η is our desired point for (4.9)+(4.10). In fact, (4.1) and the definition of M give [by (d07) and (4.8) (the first half)]

$$v_\eta \in [\Gamma \geq 2r(\eta) - \mu] \subseteq [\Gamma \geq r(\eta)]; \tag{4.13}$$

and, from this, the first part of (4.9) is clear. Combining with the second half of (4.8) and (4.11) gives

$$\alpha(\Gamma, f) - \eta^2 < f(v_\eta) \leq f(u_\eta) < \alpha(\Gamma, f) + \eta^2; \tag{4.14}$$

so, the second part of (4.9) holds too. This, again coupled with (4.11) yields (via (d07)) $d(u_\eta, v_\eta) \leq (1/\eta)2\eta^2 = 2\eta < \lambda$; wherefrom, by (4.1),

$$v_\eta \in X(u_\eta, \lambda) \subseteq [\Gamma \geq 4r(\eta) - \mu]; \tag{4.15}$$

which "improves" (4.13) above. Finally, again by (4.1) (and (d07)),

$$X(v_\eta, \lambda) \subseteq [\Gamma \geq 4r(\eta) - 2\mu] \subseteq [\Gamma \geq 2r(\eta)] \subseteq M.$$

Summing up, v_η is an interior point of M fulfilling the variational condition (4.12). This, along with the definition of the conical strong d-slope, gives (4.10); and the claim follows.

(II) Let (η_n) be a descending to zero sequence in $]0, \lambda/2\mu[$ and put $r_n = r(\eta_n)$[=the quantity of (4.8)], $n \geq 0$. Note that, by this choice, $r_n \geq 1/\eta_n$, for all n; hence $r_n \to \infty$

as $n \to \infty$. Moreover, the developments in **(I)** give us a sequence $(v_n = v_{\eta_n})$ in $\Gamma^{-1}(R_+^0)$ fulfilling

$$\Gamma(v_n) \geq r_n, \quad |f(v_n) - \alpha(\Gamma, f)| < \eta_n^2, \quad |\nabla|F(v_n, f(v_n)) \leq |\nabla|f(v_n) \leq \eta_n, \ \forall n. \tag{4.16}$$

But, from this, (4.5)-(4.7) are clear. The proof is thereby complete. □

We are now in position to give the promised answer to our coercivity question. The "hybrid" condition below is to be considered

(d08) each sequence (x_n) in $\Gamma^{-1}(R_+^0)$ for which $(f(x_n))$ converges and $|\nabla|f(x_n) \to 0$ has a subsequence (y_n) with $(\Gamma(y_n))$ bounded (in R_+).

This will be referred to as a Palais-Smale condition upon f with respect to Γ.

Theorem 4 *Suppose that (in addition) f satisfies a Palais-Smale condition with respect to Γ. Then, f (hence F as well) is Γ-coercive.*

Proof If, by absurd, this cannot happen, the relation (d06) must be true. By Theorem 3, we have promised a sequence (v_n) in $\Gamma^{-1}(R_+^0)$ with the properties (4.5)-(4.7). Combining with the imposed Palais-Smale condition it results that (v_n) must have a subsequence (y_n) with $(\Gamma(y_n))$ bounded (in R_+). On the other hand, $\Gamma(y_n) \to \infty$, by (4.5). The obtained contradiction shows that (d06) cannot be accepted; hence the conclusion. □

In particular, (d08) follows (via Lemma 3) from

(d09) each sequence (x_n) in $\Gamma^{-1}(R_+^0)$ for which $(f(x_n))$ converges and $|\nabla|F(x_n, f(x_n)) \to 0$ has a subsequence (y_n) with $(\Gamma(y_n))$ bounded (in R_+);

this will be referred to as the Palais-Smale condition upon F with respect to (Γ, f). As a consequence, we have

Theorem 5 *Suppose that (in addition) F satisfies a Palais-Smale condition with respect to (Γ, f). Then, F is necessarily Γ-coercive.*

Summing up, these "multivalued" coercivity results reduce to their "univalued" versions. So, genuine extensions of this type are to be obtained in a vectorial setting. On the other hand, a (quasi-) order extension of this result is available under the lines in D. Motreanu, V. V. Motreanu and M. Turinici [19]. We shall discuss all these in a future paper.

5 Differential versions

The obtained results are, at a first glance, "absolute" ones. For technical reasons, it would be useful having "relative" forms of them (expressed via Dini derivatives).

(A) Let $(X, ||.||)$ be a normed space; and K be some nonempty part of X. The contingent cone of K at some $z \in K$ is defined as

(e01) $\mathcal{T}(K)(z) = \{w \in X; \liminf_{\lambda \to 0} (1/\lambda)\ \mathrm{dist}(z + \lambda w, K) = 0\}$;

here, as already precise, dist(.,.) is the point to set distance attached to d (=the metric induced by $||.||$). Note that $w \in \mathcal{T}(K)(z)$ if and only if

$$\exists(\lambda_n) \subseteq R_+^0, \lambda_n \to 0,\ \exists(z_n) \subseteq K:\ w = \lim_n (1/\lambda_n)(z_n - z).$$

This also writes, for simplicity reasons

(e02) $w \in \mathrm{Lim}_n(1/\lambda_n)(K - z)$, for some sequence $(\lambda_n) \subseteq R_+^0$, $\lambda_n \to 0$.

Note that the effective meaning of this is

$$\liminf_{\lambda \to 0+} \mathrm{dist}[w, (1/\lambda)(K - z)] = 0;\ \text{ and writes: } w \in \mathrm{Lim}\sup_{\lambda \to 0+} (1/\lambda)(K - z);$$

cf. Zălinescu [32, Ch 3, Sect 3.1]. Here, for each metric space (Z, e) and each multivalued map $G : Z \to \mathcal{P}_0(X)$ with $\mathrm{Dom}(G) = Z$ we denoted

(e03) $\mathrm{Lim}\sup_{z \to c} G(z) = \{w \in X; \liminf_{z \to c}\ \mathrm{dist}(w, G(z)) = 0\}$, $c \in Z$.

Note that the conical property of $\mathcal{T}(K)(z)$ must be taken in the homogeneous sense only:

$$w \in \mathcal{T}(K)(z) \Longrightarrow \alpha w \in \mathcal{T}(K)(z), \forall \alpha > 0. \tag{5.1}$$

For, in general, $\mathcal{T}(K)(z)$ is not convex unless K is convex. Nevertheless, $\mathcal{T}(K)(z)$ is anyway closed; see Ward [30] for a thorough discussion of these facts.

(B) Now, let $(X, ||.||)$, $(Y, ||.||)$ be a couple of normed spaces; and $F : X \to \mathcal{P}_0(Y)$ be a multivalued map from X to Y. As usually, we identify F with its graph in $X \times Y$. The Dini derivative of F at $(x, y) \in F$ is the multivalued map from X to Y:

(e04) $\Delta F(x, y) = \{(u, v) \in X \times Y; (u, -v) \in \mathcal{T}(F)(x, y)\}$.

By the characterization of the contingent cone in the right hand side, it follows that $(u, v) \in \Delta F(x, y)$ if and only if

$$\exists(\lambda_n) \subseteq R_+^0, \lambda_n \to 0, \exists((x_n, y_n)) \subseteq F:\ u = \lim_n (1/\lambda_n)(x_n - x),\ v = \lim_n (1/\lambda_n)(y - y_n).$$

Putting $w_n = (1/\lambda_n)(x_n - x)$, we have $x_n = x + \lambda_n w_n$ (for all n); whence

$$(y_n \in F(x_n) = F(x + \lambda_n w_n), n \geq 0);\ \text{with } \lambda_n \to 0, w_n \to u.$$

So, for a fixed $u \in X$, one has $v \in \Delta F(x, y)(u)$ if and only if

(e05) $\exists(\lambda_n) \subseteq R_+^0$, $\lambda_n \to 0$, $\exists(w_n) \subseteq X$ $w_n \to u$, $\exists(y_n) \subseteq Y$:
$(y_n \in F(x + \lambda_n w_n); n \geq 0)$ and $v = \lim_n(1/\lambda_n)(y - y_n)$.

This also writes, for simplicity

(e06) $\exists(\lambda_n) \subseteq R_+^0$, $\lambda_n \to 0$, $\exists(w_n) \subseteq X$ $w_n \to u$: $v \in \mathrm{Lim}_n(1/\lambda_n)(y - F(x + \lambda_n w_n))$;

or equivalently (by the definition above)

$$v \in \operatorname{Lim\,sup}_{\substack{w\to u\\ \lambda\to 0+}} (1/\lambda)(y - F(x + \lambda w));$$

see, for instance, Aubin and Frankowska [3, Ch 5, Sect 5.1]. The multivalued map $u \mapsto \Delta F(x, y)(u)$ exists at least in $u = 0$; precisely,

$$(0 \in \mathrm{Dom}(\Delta F(x, y)) \text{ and) } 0 \in \Delta F(x, y)(0). \tag{5.2}$$

Moreover, it is positively homogeneous over its domain

$$\Delta F(x, y)(\alpha u) = \alpha \Delta F(x, y)(u), \forall \alpha > 0; \tag{5.3}$$

where, by convention, $\alpha\emptyset = \emptyset$, $\forall \alpha > 0$. Unfortunately, $\mathrm{Dom}(\Delta F(x, y)) = \{0\}$ cannot be avoided; i.e.: for many nonzero directions $u \in X$ the limit in (e06) may be empty.

(C) Let $F : X \to \mathcal{P}_0(R)$ be a multivalued map from X to R. Define, for $(x, y) \in F$

(e07) $\Delta_\infty F(x, y)(u) = \sup[\Delta F(x, y)(u)]$, $u \in X$.

[As usually, the supremum of the empty set is $(-\infty)$]. The obtained map $u \mapsto \Delta_\infty F(x, y)(u)$ is well defined over all of X, with values in $\widetilde{R} := R \cup \{-\infty, \infty\}$. Hence, the convention below is meaningful, for each $(x, y) \in F$:

(e08) $|\delta|F(x, y) = \max\{0, \delta F(x, y) := \sup\{\Delta_\infty F(x, y)(u); u \in X_{(1)}\}\}$.

[Here, $X_{(1)} := \{x \in X; ||x|| = 1\}$ is the boundary of the unit sphere in X]. It will be referred to as the *differential* strong d-slope of F at (x, y). A natural question is to establish the connection between this indicator and the one introduced in Section 4. The following answer is to be noted.

Lemma 4 *Let $(x, y) \in F$ be arbitrary fixed. Then*

$$\delta F(x, y) \le \nabla F(x, y); \text{ hence } |\delta|F(x, y) \le |\nabla|F(x, y). \tag{5.4}$$

Proof We firstly show that, for each $u \in X_{(1)}$, one has $\Delta_\infty F(x, y)(u) \le \nabla F(x, y)$. The case of $\Delta F(x, y)(u) = \emptyset$ is clear; so, it remains to discuss the alternative $\Delta F(x, y)(u) \ne \emptyset$. Each $v \in \Delta F(x, y)(u)$ has the representation (e06); where $(\lambda_n) \subseteq R^0_+$, $\lambda_n \to 0$ and $(w_n) \subseteq X$, $w_n \to u$. Denote $(\alpha_n = ||w_n||; n \ge 0)$ [hence $\alpha_n \to 1$] and $(z_n = w_n/\alpha_n; n \ge 0)$, $(\mu_n = \lambda_n\alpha_n; n \ge 0)$ [hence $\mu_n \to 0$]. By the precise definition, we have $v \in \mathrm{Lim}_n(\alpha_n/\mu_n)(y - F(x + \mu_n z_n))$; and this, along with $\alpha_n \to 1$, gives $v \in \mathrm{Lim}_n(1/\mu_n)(y - F(x + \mu_n z_n))$, where $(\mu_n) \subseteq R^0_+$, $\mu_n \to 0$, $(z_n) \subseteq X_{(1)}$, $z_n \to u$. Let $\varepsilon > 0$ be arbitrary fixed. From the very choice of (μ_n), there must be some $n(\varepsilon)$ with $n \ge n(\varepsilon) \Longrightarrow \mu_n < \varepsilon$. Combining with the properties of (z_n) gives

$$(\forall n \ge n(\varepsilon)) : \quad \frac{1}{\mu_n}[y - F(x + \mu_n z_n)] \subseteq \{\frac{y - F(w)}{d(x, w)};\ 0 < d(x, w) < \varepsilon\}.$$

Denote by $A(\varepsilon)$ the supremum in the right hand side. We have (from the above) $(1/\mu_n)[y - F(x + \mu_n z_n)] \le A(\varepsilon)$, for all $n \ge n(\varepsilon)$; wherefrom (passing to limit as $n \to \infty$) $v \le A(\varepsilon)$.

As v is arbitrarily fixed in $\Delta F(x,y)(u)$, one gets $\Delta F(x,y)(u) \le A(\varepsilon)$; hence (by definition) $\Delta_\infty F(x,y)(u) \le A(\varepsilon)$. Passing to supremum over $u \in X_{(1)}$, we derive $\delta F(x,y) \le A(\varepsilon)$; and from this, the expected relation follows by taking the infimum upon $\varepsilon > 0$. □

We may now give the announced differential version of the results in Section 4. Let the function $\Gamma : X \to R_+$ be as in (d01)+(d02); and the multivalued map $F : X \to \mathcal{P}_0(R)$ (from X to R) be as in (d03). The differential type "hybrid" condition below is to be considered:

(e09) each sequence (x_n) in $\Gamma^{-1}(R^0_+)$ for which $(f(x_n))$ converges and $|\delta|F(x_n, f(x_n)) \to 0$ has a subsequence (u_n) with $(\Gamma(u_n))$ bounded.

(Here, $f = \inf[F]$ is the inf-selection of F). This will be referred to as the differential Palais-Smale condition upon F with respect to (Γ, f). Note that, by Lemma 4 above, (e09) implies (d09). This, along with Theorem 5, gives the following differential coercivity criterion:

Theorem 6 *Assume (in addition) that F fulfills a differential Palais-Smale condition with respect to (Γ, f). Then, F is necessarily Γ-coercive.*

(D) An interesting particular version of the constructions above may be given along the following lines. Let $(X, ||.||)$, $(Y, ||.||)$ be a couple of normed spaces; and $F : X \to \mathcal{P}_0(Y)$ be a multivalued map from X to Y. For each $(x,y) \in F$, let us introduce a relation $\Theta F(x,y)$ on $X \times Y$ as: for $u \in X$, we say that $v \in \Theta F(x,y)(u)$ if and only if

(e10) $\exists(\lambda_n) \subseteq R^0_+$, $\lambda_n \to 0$, $\exists(y_n) \subseteq Y$:
$(y_n \in F(x + \lambda_n u); n \ge 0)$ and $v = \lim_n (1/\lambda_n)(y - y_n)$.

This also writes, for simplicity

(e11) $\exists(\lambda_n) \subseteq R^0_+$, $\lambda_n \to 0$: $w \in \mathrm{Lim}_n (1/\lambda_n)(y - F(x + \lambda_n u))$;

or equivalently (by the preceding definitions above)

$$w \in \mathop{\mathrm{Lim\,sup}}_{\lambda \to 0+} (1/\lambda)(y - F(x + \lambda u)).$$

By definition, $\Theta F(x,y)$ will be referred to as the local Dini derivative for F at $(x,y) \in F$. As before, the multivalued map $u \mapsto \Theta F(x,y)(u)$ exists at least in $u = 0$ and is positively homogeneous over its domain; cf (5.2)+(5.3). Moreover, we have (by these very conventions)

$$\Theta F(x,y) \subseteq \Delta F(x,y), \text{ for all } (x,y) \in F. \tag{5.5}$$

The inclusion may be strict, as simple examples show.

In particular, let $F : X \to \mathcal{P}_0(R)$ be a multivalued map from X to R. Define, for $(x,y) \in F$

(e12) $\Theta_\infty F(x,y)(u) = \sup[\Theta F(x,y)(u)]$, $u \in X$.

The obtained map $u \mapsto \Theta_\infty F(x,y)(u)$ is well defined over all of X, with values in $\widetilde{R}$. In addition, we have (by (5.5)) at each $(x,y) \in F$

$$\Theta_\infty F(x,y)(u) \le \Delta_\infty F(x,y)(u), \text{ for all } u \in X. \tag{5.6}$$

Finally, let us define for each $(x,y) \in F$

(e13) $|\theta|F(x,y) = \max\{0, \theta F(x,y) := \sup\{\Theta_\infty F(x,y)(u); u \in X_{(1)}\}\}$.

This will be referred to as the *local differential* strong d-slope of F at (x,y). By (5.6) above, it is clear that, for each $(x,y) \in F$

$$\theta F(x,y) \leq \delta F(x,y);\ \text{hence } |\theta|F(x,y) \leq |\delta|F(x,y). \tag{5.7}$$

Having these precise, let the function $\Gamma : X \to R_+$ be as in (d01)+(d02); and the multivalued map $F : X \to \mathcal{P}_0(R)$ (from X to R) be as in (d03). The differential type "hybrid" condition below is to be considered:

(e14) each sequence (x_n) in $\Gamma^{-1}(R_+^0)$ for which $(f(x_n))$ converges and $|\theta|F(x_n, f(x_n)) \to 0$ has a subsequence (u_n) with $(\Gamma(u_n))$ bounded.

This will be referred to as the local differential Palais-Smale condition upon F with respect to (Γ, f). Note that by (5.7) above, (e14) implies (e09). Combining with Theorem 6 gives the following differential coercivity criterion:

Theorem 7 *Assume (in addition) that F fulfills a local differential Palais-Smale condition with respect to (Γ, f). Then, F is necessarily Γ-coercive.*

(E) The obtained result is for the moment a particular version of the preceding one (Theorem 6). So, it is legitimate to ask whether this (logical) inclusion may be reversed. A concrete circumstance yielding such a relation it to be described as

(e15) F is locally Lipschitz: $\forall x \in X$, $\exists \varepsilon(x) > 0$, $\exists \nu(x) > 0$: $F(x') \subseteq F(x") + \nu(x)||x' - x"||[-1,1]$, $\forall x', x" \in X(x, \varepsilon(x))$.

The following auxiliary fact will clarify this.

Lemma 5 *Let the multivalued map $F : X \to \mathcal{P}_0(R)$ (from X to R) be locally Lipschitz. Then, for each $(x,y) \in F$,*

$$\Delta F(x,y) \subseteq \Theta F(x,y);\ \textit{hence } \Delta F(x,y) = \Theta F(x,y) \tag{5.8}$$

$$\Delta_\infty F(x,y) = \Theta_\infty F(x,y),\ \delta F(x,y) = \theta F(x,y),\ |\delta|F(x,y) = |\theta|F(x,y). \tag{5.9}$$

Proof Let $(u,v) \in \Delta F(x,y)$ be arbitrary fixed; hence $v \in \Delta F(x,y)(u)$ has the representation (e05). Take some rank $p = p[x]$ according to

(e16) $\lambda_n||w_n|| \leq \varepsilon(x)$, $\lambda_n||u|| \leq \varepsilon(x)$, for all $n \geq p$.

By (e15), the sequence $(y_n; n \geq 0)$ appearing there admits the decomposition (for all $n \geq p$): $y_n = y'_n + \nu(x)\lambda_n||w_n - u||z_n$; where $y'_n \in F(x + \lambda_n u)$, $z_n \in [-1,1]$, for all such n. So,

$$v = \lim_n (1/\lambda_n)(y - y'_n) - \lim_n(\nu(x)||w_n - u||z_n) = \lim_n (1/\lambda_n)(y - y'_n);$$

which tells us that $v \in \Theta F(x,y)(u)$; or, equivalently: $(u,v) \in \Theta F(x,y)$. This proves (5.8); hence, (5.9) as well. □

In other words, (e09) and (e14) are identical under (e15). On the other hand, (e15) yields the upper semicontinuous condition (c08); which in turn gives (as precise in Section 3) the closed (graph) condition (c05) whenever F has nonempty closed values (cf. (c07)). This, along with (c02)+(c05) $\Longrightarrow$ (c03) (cf. Section 3) yields the following coercivity criterion:

Theorem 8 *Let the function* $\Gamma : X \to R_+$ *be as in (d01)+(d02); and the multivalued map* $F : X \to \mathcal{P}_0(R)$ *(from* X *to* R*) be such that (c02)+(c07)+(e15) hold, as well as (e09) (or equivalently, (e14)). Then,* F *is* Γ*-coercive.*

In particular, when (c07) is taken in the stronger sense

(e17) F is (nonempty) compact valued: $(\emptyset \neq)F(x)$=compact, $\forall x \in X$

Theorem 8 is just the main result in Kristaly and Varga [17]. Some "conical" extensions of these results are directly obtainable under the lines in Motreanu and Turinici [21]; further aspects will be delineated elsewhere.

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