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Regularization of Certain Divergent Series of Polynomials

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Abstract. We investigate the generalized convergence and sums of series of the form $\sum_{n\geq 0} a_n \boldsymbol{T}^n P(x)$, where $P \in \mathbb{R}[x]$, $a_n \in \mathbb{R}$, $\forall n \geq 0$, and $\boldsymbol{T} : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator that commutes with the differentiation $\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x]$.

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1 The main result

We consider series of the form

$$\sum_{n\geq 0} a_n \boldsymbol{T}^n P(x), \tag{$\dagger$}$$

where $P \in \mathbb{R}[x]$, and $\boldsymbol{T} : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator such that

$$\boldsymbol{T} D = D \boldsymbol{T}, \tag{$*$}$$

where D is the differentiation operator $D = \frac{d}{dx}$. The condition $(*)$ is equivalent with the translation invariance of $\boldsymbol{T}$, i.e.,

$$\boldsymbol{T}\boldsymbol{U}^h = \boldsymbol{U}^h \boldsymbol{T}, \;\; \forall h \in \mathbb{R}, \tag{I}$$

where $\boldsymbol{U}^h : \mathbb{R}[x] \to \mathbb{R}[x]$ is the translation operator

$$\mathbb{R}[x] \ni p(x) \mapsto p(x+h) \in \mathbb{R}[x].$$

For simplicity we set $\boldsymbol{U} := \boldsymbol{U}^1$. Clearly $\boldsymbol{U}^h \in \mathcal{O}$ so a special case of the series $(\dagger)$ is the series

$$\sum_{n\geq 0} a_n \boldsymbol{U}^{nh} P(x) = \sum_{n\geq 0} a_n P(x+nh), \;\; h \in \mathbb{R}, \tag{$\ddagger_h$}$$

which is typically divergent.

We denote by $\mathcal{O}$ the $\mathbb{R}$-algebra of translation invariant operators. We have a natural map

$$\mathcal{Q} : \mathbb{R}[[t]] \to \mathcal{O}, \;\; \mathbb{R}[[t] \ni \sum_{n\geq 0} c_n \frac{t^n}{n!} \mapsto \sum_{n\geq 0} \frac{c_n}{n!} D^n.$$

It is known (see [1, Prop. 3.47]) that this map is an isomorphism of rings. We denote by $\boldsymbol{\sigma}$ the inverse of $\mathcal{Q}$

$$\boldsymbol{\sigma} : \mathcal{O} \to \mathbb{R}[[t]],\ \ \mathcal{O} \ni \boldsymbol{T} \mapsto \boldsymbol{\sigma}_{\boldsymbol{T}} \in \mathbb{R}[[t]].$$

For $\boldsymbol{T} \in \mathcal{O}$ we will refer to the formal power series $\boldsymbol{\sigma}_{\boldsymbol{T}}$ as the *symbol* of the operator $\boldsymbol{T}$. More explicitely

$$\boldsymbol{\sigma}_{\boldsymbol{T}}(t) = \sum_{n\geq 0} \frac{c_n(\boldsymbol{T})}{n!} t^n,\ \ c_n(\boldsymbol{T}) = (\boldsymbol{T}x^n)|_{x=0} \in \mathbb{R}.$$

We denote by $\mathbb{N}$ the set of nonnegative integers, and by **Seq** the vector space of real sequences, i.e., maps $a : \mathbb{N} \to \mathbb{R}$. Let $\mathbf{Seq}^c$ the vector subspace of **Seq** consisting of all convergent sequences.

A *generalized notion of convergence*[1] or *regularization method* is a pair $\mu = ({}^{\mu}\lim, \mathbf{Seq}_{\mu})$, where

- $\mathbf{Seq}_{\mu}$ is a vector subspace of **Seq** containing $\mathbf{Seq}^c$ and,
- ${}^{\mu}\lim$ is a linear map

$$^{\mu}\lim : \mathbf{Seq}_{\mu} \to \mathbb{R},\ \ \mathbf{Seq}_{\mu} \ni a \mapsto {}^{\mu}\lim_{n} a(n) \in \mathbb{R}$$

such that for any $a \in \mathbf{Seq}^c$ we have

$$^{\mu}\lim a = \lim_{n\to\infty} a(n).$$

The sequences in $\mathbf{Seq}_{\mu}$ are called *μ-convergent* and ${}^{\mu}\lim$ is called the *μ-limit*. To any sequence $a \in \mathbf{Seq}$ we associate the sequence $\boldsymbol{S}[a]$ of partial sums

$$\boldsymbol{S}[a](n) = \sigma_{k=0}^{n} a(k). \tag{1.1}$$

A series $\sum_{n\geq 0} a(n)$ is said to by μ-convergent if the sequence $\boldsymbol{S}[a]$ is μ-convergent. We set

$$^{\mu}\sum_{n\geq 0} a(n) := {}^{\mu}\lim_{n} \boldsymbol{S}[a](n).$$

We say that ${}^{\mu}\sum_{n\geq 0} a(n)$ is the *μ-sum of the series*. The regularization method is said to be *shift invariant* if it satisfies the additional condition

$$^{\mu}\sum_{n\geq 0} a(n) = a(0) + {}^{\mu}\sum_{n\geq 1} a(n). \tag{1.2}$$

We refer to the classic [3] for a large collection of regularization methods.

For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ we set

$$[x]_k := \begin{cases} \prod_{i=0}^{k-1}(x-i), & k \geq 1 \\ 1, & k = 0, \end{cases},\ \binom{x}{k} := \frac{[x]_k}{k!}.$$

We can now state the main result of this paper.

[1]Hardy refers to such a notion of convergence as convergence in some ‘Pickwickian’ sense.

Theorem 1.1. *Let μ be a regularization method, $\boldsymbol{T} \in \mathcal{O}$ and $f(t) = \sum_{n\geq 0} a_n t^n \in \mathbb{R}[[t]]$. Set $c := c_0(\boldsymbol{T}) = \boldsymbol{T}1$. Suppose that f is μ*-regular at $t = c$, *i.e.,*

$$\text{for every } k \in \mathbb{N} \text{ the series } \sum_{n\geq 0} a_n [n]_k \, c^{n-k} \text{ is } \mu\text{-convergent.} \tag{μ}$$

We denote by $f^{(k)}(c)_\mu$ its μ-sum

$$f^{(k)}(c)_\mu \, := {}^{\mu}\sum_{n\geq 0} a_n [n]_k \, c^{n-k}.$$

Then for every $P \in \mathbb{R}[x]$ the series $\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x)$ is μ-convergent and its μ-sum is

$${}^{\mu}\sum a_n (\boldsymbol{T}^n P)(x) = f(\boldsymbol{T})_\mu P\,(x),$$

where $f(\boldsymbol{T})_\mu \in \mathcal{O}$ is the operator

$$f(\boldsymbol{T})_\mu := \sum_{n\geq 0} \frac{f^{(k)}(c)_\mu}{k!} (\boldsymbol{T} - c)^k. \tag{1.3}$$

Proof. Set $\boldsymbol{R} := \boldsymbol{T} - c$ and let $P \in \mathbb{R}[x]$. Then

$$\boldsymbol{R} = \sum_{n\geq 1} \frac{c_n(\boldsymbol{T})}{n!} D^n$$

so that

$$\boldsymbol{R}^n P = 0, \ \ \forall n > \deg P. \tag{1.4}$$

In particular this shows that $f(\boldsymbol{T})_\mu$ is well defined. We have

$$a_n \boldsymbol{T}^n P = a_n (c + \boldsymbol{R})^n P = a_n \sum_{k=0}^{n} \binom{n}{k} c^{n-k} \boldsymbol{R}^k P = \sum_{k=0}^{\deg P} \binom{n}{k} c^{n-k} \boldsymbol{R}^k P.$$

At the last step we used (1.4) and the fact that

$$\binom{n}{k} = 0, \ \text{ if } \ k > n.$$

This shows that the formal series $\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x)$ can be written as a *finite* linear combination of formal series

$$\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x) = \sum_{k=0}^{\deg P} \frac{\boldsymbol{R}^k P(x)}{k!} \left(\sum_{n\geq 0} a_n [n]_k \, c^{n-k} \right).$$

From the linearity of the μ-summation operator we deduce

$${}^{\mu}\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x) = \sum_{k=0}^{\deg P} \frac{\boldsymbol{R}^k P(x)}{k!} \left({}^{\mu}\sum_{n\geq 0} a_n [n]_k \, c^{n-k} \right)$$

$$= \left(\sum_{k=0}^{\deg P} \frac{f^{(k)}(c)_\mu}{k!} \boldsymbol{R}^k \right) P(x) = f(\boldsymbol{T})_\mu P\,(x)$$

□

2 Some applications

To describe some consequences of Theorem 1.1 we need to first describe some classical facts about regularization methods.

For any sequence $a \in \mathbf{Seq}$ we denote by $\boldsymbol{G}_a(t) \in \mathbb{R}[[t]]$ its generating series. We regard the partial sum construction $\boldsymbol{S}$ in (1.1) as a linear operator $\boldsymbol{S} : \mathbf{Seq} \to \mathbf{Seq}$. Observe that

$$\boldsymbol{G}_{\boldsymbol{S}[a]}(t) = \frac{1}{1-t} \boldsymbol{G}_a(t).$$

We say that a regularization method $\mu_1 = ({}^{\mu_1}\lim, \mathbf{Seq}_{\mu_1})$ is stronger than the regularization method $\mu_0 = ({}^{\mu_1}\lim, \mathbf{Seq}_{\mu_0})$, and we write this $\mu_0 \prec \mu_1$, if

$$\mathbf{Seq}_{\mu_0} \subset \mathbf{Seq}_{\mu_1} \;\; \text{and} \;\; {}^{\mu_1}\lim_n a(n) = {}^{\mu_0}\lim_n a(n), \;\; \forall a \in \mathbf{Seq}_{\mu_0}.$$

The *Abel regularization method*[2] A is defined as follows. We say that a sequence a is A convergent if

- the radius of convergence of the series $\sum_{n\geq 0} a_n t^n$ is at least 1 and
- the function $t \mapsto (1-t)\sum_{n\geq 0} a_n t^n$ has a finite limit as $t \to 1^-$.

Hence

$${}^{A}\lim a(n) = \lim_{t\to 1^-} (1-t) \sum_{n\geq 0} a_n t^n,$$

and $\mathbf{Seq}_A$ consists of sequence for which the above limit exists and it is finite. Using (2) we deduce that a series $\sum_{n\geq 0} a(n)$ is A-convergent if and only if the limit

$$\lim_{t\to 1^-} \sum_{n\geq 0} a_n t^n$$

exists and it is finite. We have the following immediate result.

Proposition 2.1. *Suppose that $f(z)$ is a holomorphic function defined in an open neighborhood of the set $\{1\} \cup \{|z|\} \subset \mathbb{C}$. If $\sum_{n\geq 0} a_n z^n$ is the Taylor series expansion of f at $z = 0$ then the corresponding formal power series $[f] = \sum_{n\geq 0} a_n t^n$ is A-regular at $t = 1$,*

$$[f]^{(k)}(1)_A = f^k(1),$$

and the series

$$[f](r)_A = \sum_k \frac{[f]^{(k)}(1)_A}{k!} r^k$$

coincides with the Taylor expansion of f at $z = 1$, and it converges to $f(1+r)$.

[2]This was apparently known and used by Euler.

Corollary 2.2. *Suppose that $f(z)$ is a holomorphic function defined in an open neighborhood of the set $\{1\}\cup\{|z|\}\subset\mathbb{C}$ and $\sum_{n\geq 0} a_n z^n$ is the Taylor series expansion of f at $z=0$. Then for every $\boldsymbol{T}$ in $\mathcal{O}$ such that $c_0(\boldsymbol{T})=1$, any $P\in\mathbb{R}[x]$, and any $x\in\mathbb{R}$ we have*

$$ {}^{A}\sum_{n} a_n \boldsymbol{T}^n P(x) = \sum_{k\geq 0} \frac{f^k(1)}{k!}(\boldsymbol{T}-1)^k P(x). \qquad \square $$

Let $k\in\mathbb{N}$. A sequence $a\in\mathbf{Seq}$ is said to be *C_k-convergent* (or *Cesàro convergent of order k*) if the limit

$$ \lim_{n\to\infty} \frac{\boldsymbol{S}^k[a](n)}{\binom{n+k}{k}} $$

exists and it is finite. We denote this limit by ${}^{C_k}\lim a(n)$. A series $\sum_{n\geq 0} a(n)$ is said to be C_k-convergent if the sequence of partial sums $\boldsymbol{S}[a]$ is C_k convergent. Thus the C_k-sum of this series is

$$ {}^{C_k}\sum_{n\geq 0} a(n) = \lim_{n\to\infty} \frac{\boldsymbol{S}^{k+1}[a](n)}{\binom{n+k}{k}}. $$

More explicitly, we have (see [3, Eq.(5.4.5)])

$$ {}^{C_k}\sum_{n\geq 0} a(n) = \lim_{n\to\infty} \frac{1}{\binom{n+k}{k}} \left(\sum_{\nu=0}^{n} \binom{\nu+k}{k} a(n-\nu) \right) $$

Hence

$$ {}^{C_k}\sum_{n\geq 0} a(n) \Longleftrightarrow \boldsymbol{S}^{k+1}[a](n) \sim A\binom{n+k}{k} \sim A\frac{n^k}{k!}, $$

where

$$ a\sim b \Longleftrightarrow \lim_{n\to\infty}\frac{a(n)}{b(n)} = 1, $$

if $a(n), b(n)\neq 0$, for $n\gg 0$.

The C_0 convergence is equivalent with the classical convergence and it is known (see [3, Thm. 43, 55]) that

$$ C_k \prec C_{k'} \prec A, \;\; \forall k<k'. $$

Given this fact, we define a sequence to be C-convergent (Cesàro convergent) if it is C_k-convergent for some $k\in\mathbb{N}$. Note that $C\prec A$. Both the C and A methods are shift invariant, i.e., they satisfy the condition (1.2).

We want to comment a bit about possible methods of establishing C-convergence. To formulate a general strategy we need to introduce a classical notation. More precisely, if $f(t)=\sum_{n\geq 0} a_n t^n$ is a formal power series we let $[t^n]f(t)$ denote the coefficient of t^n in this power series, i.e. $[t^n]f(t)=a_n$.

Let $f(t)=\sum_{n\geq 0} a_n t^n$. Then the series $\sum_{n\geq 0} a_n t^n$ C-converges to A if and only if there exists a nonnegative *real* number α such that

$$ [t^n]\Big((1-t)^{-(\alpha+1)} f(t) \Big) \sim A\frac{n^\alpha}{\Gamma(\alpha+1)}, $$

where Γ is Euler's Gamma function. For a proof we refer to [3, Thm. 43].

Definition 2.3. We say that a power series $f(t)=\sum_{n\geq 0}a_nt^n$ is *Cesàro convenient (or C-convenient) at* 1 *if the following hold.*

(i) The radius of convergences of the series is ≥ 1

(ii) The function f is regular at $z=1$ and has finitely many singularities $\zeta_1,\dots,\zeta_\nu\neq 1$ on the unit circle $\{|z|=1\}$.

(iii) There exist $\varepsilon>0$ and $\theta\in(0,\frac{\pi}{2})$ such that f admits a continuation to the dimpled disk

$$\Delta_{\varepsilon,\theta}:=\left\{ z\in\mathbb{C};\ \ |z|<1+\varepsilon,\ \ \arg\Bigl(\frac{z}{\zeta_j}-1\Bigr)>\theta,\ \ \forall j=1,\dots,\nu\right\}.$$

(iv) For every singular point ζ_j there exists a positive integer m_j such that

$$f(z)=O\bigl((z-\zeta_j)^{-m_j}\bigr)\ \text{ as } z\to\zeta_j,\ z\in\Delta.$$

□

The results in [2, Chap. VI] implies that the collection $\mathcal{R}_C$ of C-convenient power series is a ring satisfying

$$f\in\mathcal{R}_C\Longleftrightarrow\frac{df}{dt}\in\mathcal{R}_C.$$

Invoking [2, Thm VI.5] we deduce the following useful consequence.

Corollary 2.4. *Let $f\in\mathbb{R}[[t]]$ be a power series C-convenient at* 1*. Then f is C-regular at* 1 *and*

$$f^k(1)_C=f^{(k)}(1)_A=f^{(k)}(1).$$ □

Using [2, VII.7] we obtain the following useful result.

Corollary 2.5. *(a) The power series*

$$(1+t)^{-m}=\sum_{n\geq 0}\binom{n+m-1}{n}(-t)^n,\ \ m\geq 1,\ \ \log(1+t)=\sum_{n\geq 1}(-1)^{n+1}\frac{t^n}{n}$$

are C-regular at 1.
(b) If $f(z)$ is an algebraic function defined on the unit disk $|z|<1$ and regular at $z=1$ then the Taylor series of f at $z=0$ is C-regular at 1.

Recall that the Cauchy product of two sequences $a,b\in\mathbf{Seq}$ is the sequence $a*b$,

$$a*b(n)=\sum_{i=0}^{n}a(n-i)b(i),\ \ \forall n\in\mathbb{N}.$$

A regularization method is said to be *multiplicative* if

$$ {}^{\mu}\sum_n a*b(n) = \left({}^{\mu}\sum_n a(n) \right)\left({}^{\mu}\sum_n b(n) \right), $$

for any μ-convergent series $\sum_{n\geq 0} a(n)$ and $\sum_{n\geq 0} b(n)$. The results of [3, Chap.X] show that the C and A methods are multiplicative.

For any regularization method μ and $c \in \mathbb{R}$ we denote by $\mathbb{R}[[t]]_\mu$ the set of series that are μ-regular at $t = 1$.

Proposition 2.6. *Let μ be a multiplicative regularization method. Then $\mathbb{R}[[t]]_\mu$ is a commutative ring with one and we have the product rule*

$$ (f\cdot g)^{(n)}(1)_\mu = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(1)_\mu \cdot g^{(n-k)}(1)_\mu. $$

Moreover, if $\boldsymbol{T} \in \mathcal{O}$ is such that $c_0(T) = 1$ then the map

$$ \mathbb{R}[[t]]_\mu \ni f \mapsto f(\boldsymbol{T})_\mu \in \mathcal{O} $$

is a ring morphism.

Proof. The product formula follows from the iterated application of the equalities

$$ D_t(fg) = (D_t f)g + f(D_t g),\ \ (fg)(1)_\mu = f(1)_\mu \cdot g(1)_\mu,\ \ f'(1)_\mu = (D_t f)(1)_\mu, $$

where $D_t : \mathbb{R}[[t]] \to \mathbb{R}[[t]]$ is the formal differentiation operator $\frac{d}{dt}$. The last statement is an immediate application of the above product rule. □

Remark 2.7. The inclusion $\mathbb{R}[[t]]_C \subset \mathbb{R}[[t]]_A$ is strict. For example, the power series

$$ f(z) = e^{1/(1+z)} $$

satisfies the assumption of Proposition 2.1 so that the associated formal power series $[f]$ is A-regular at 1. On the other hand, the arguments in [3, §5.12] show that $[f]$ is not C-regular at 1. □

Consider the translation operator $\boldsymbol{U}^h \in \mathcal{O}$. From Taylor's formula

$$ p(x+h) = \sum_{n\geq 0} \frac{h^n}{n!} D^n p(x) $$

we deduce that

$$ \boldsymbol{\sigma}_{\boldsymbol{U}^h}(t) = e^{th}. $$

Set $\Delta_h := \boldsymbol{U}^h - 1$. Using Corollary 2.5 and Theorem 1.1 we deduce the following result.

Corollary 2.8. *For any $P \in \mathbb{R}[x]$ we have*

$$ {}^C\sum_{n\geq 0}(-1)^n P(x+nh) = \frac{1}{2}\left(\sum_{n\geq 0}\frac{(-1)^n}{2^n}\Delta_h^n\right)P(x). \tag{2.1}$$

Observe that

$$\left(1+\frac{1}{2}\Delta_h\right)\left(\sum_{n\geq 0}\frac{(-1)^n}{2^n}\Delta_h^n\right) = 1$$

so that $\frac{1}{2}\sum_{n\geq 0}\frac{(-1)^n}{2^n}\Delta_h^n$ is the inverse of the operator $2+\Delta_h$. We thus have

$$ {}^C\sum_{n\geq 0}(-1)^n P(x+nh) = (2+\Delta_h)^{-1}P(x) = (1+\boldsymbol{U}^h)^{-1}P(x). \tag{2.2}$$

Remark 2.9. Here is a heuristic explanation of the equality (2.2) assuming the Cesàro convergence of the series $\sum_{n\geq 0}(-1)^n P(x+nh)$. Denote by $S(x)$ the Cesàro sum of this series. Then

$$S(x+h) = {}^C\sum_{n\geq 0}(-1)^n P\big(\,x+(n+1)h\,\big)$$

$$\stackrel{(1.2)}{=} -{}^C\sum_{n\geq 0}(-1)^n P(x+h) + P(x) = -S(x) + P(x).$$

Hence

$$S(x+h)+S(x) = P(x),\ \ \forall x\in\mathbb{R}.$$

If we knew that $S(x)$ is a polynomial we would then deduce

$$S(x) = (1+\boldsymbol{U}^h)^{-1}P(x). \qquad \square$$

The inverse of $1+\boldsymbol{U}^h$ can be explicitly expressed using Euler numbers and polynomials, [4, Eq. (14), p.134]. The Euler numbers E_k are defined by the Taylor expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t+e^{-t}} = \sum_{k\geq 0}\frac{E_k}{k!}t^k.$$

Since $\cosh t$ is an even function we deduce that $E_k = 0$ for odd k. They satisfy the recurrence relation

$$E_n + \binom{n}{2}E_{n-2} + \binom{n}{4}E_{n-4} + \cdots = 0,\ \ n\geq 2. \tag{2.3}$$

Here are the first few Euler numbers.

n	0	2	4	6	8	10	12	14	16
E_n	1	–1	5	–61	1,385	–50,521	2,702,765	-199,360,981	19,391,512,145

Then

$$\frac{1}{1+\boldsymbol{U}^h} = \frac{\boldsymbol{U}^{-\frac{h}{2}}}{\boldsymbol{U}^{\frac{h}{2}} + \boldsymbol{U}^{-\frac{h}{2}}} = \frac{\boldsymbol{U}^{-\frac{1}{2}}}{e^{\frac{D}{2}} + e^{-\frac{D}{2}}} = \frac{1}{2}\boldsymbol{U}^{-\frac{h}{2}} \frac{1}{\cosh \frac{hD}{2}} = \frac{1}{2}\boldsymbol{U}^{-\frac{h}{2}} \sum_{k\geq 0} \frac{E_k h^k}{2^k k!} D^k.$$

Hence

$$^C\sum_{n\geq 0}(-1)^n P(x+nh) = \frac{1}{2}\sum_{k\geq 0} \frac{E_k h^k}{2^k k!} P^{(k)}\left(x - \frac{h}{2}\right). \tag{2.4}$$

When $P(x) = x^m$, $h = 1$, we have

$$^C\sum_{n\geq 0}(-1)^n (x+n)^m = \frac{1}{2}\sum_{k\geq 0} \binom{m}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{m-k}. \tag{2.5}$$

Setting $x = 0$ and using the equality $E_{2j+1} = 0$, $\forall j$ we conclude that

$$^C\sum_{n\geq 0}(-1)^n n^m = \frac{1}{2^{m+1}}\sum_{k\geq 0}(-1)^{m-k} E_k \binom{m}{k} = \frac{(-1)^m}{2^{m+1}}\sum_{k\geq 0} E_{2k}\binom{m}{2k}. \tag{2.6}$$

Using (2.3) we deduce that when m is even, $m = 2m'$, $m' > 0$ we have

$$^C\sum_{n\geq 0}(-1)^n n^{2m'} = 0. \tag{2.7}$$

For example

$$1 - 1 + 1 - 1 + \cdots \overset{C}{=} \frac{1}{2}, \tag{$\dagger_0$}$$

$$-1 + 2 - 3 + 4 - \cdots \overset{C}{=} -\frac{1}{4}, \tag{$\dagger_1$}$$

$$-1 + 2^3 - 3^3 + 4^3 - \cdots \overset{C}{=} \frac{1}{8}, \tag{$\dagger_3$}$$

$$-1^5 + 2^5 - 3^5 + 4^5 - \cdots \overset{C}{=} -\frac{1}{4}. \tag{$\dagger_5$}$$

When $P(x) = \binom{x}{m}$, $x = 0$, $h = 1$ then it is more convenient to use (2.1) because

$$\Delta\binom{x}{k} = \binom{x}{k-1}, \quad \forall k, x.$$

We deduce

$$^C\sum_{n\geq 0}(-1)^n \binom{n}{m} = \frac{1}{2}\sum_{k=1}^{m} \frac{(-1)^k}{2^k}\binom{0}{m-k} = \frac{(-1)^m}{2^{m+1}}. \tag{2.8}$$

Example 2.10. Consider the translation invariant operator

$$\boldsymbol{T} : \mathbb{R}[x] \to \mathbb{R}[x], \;\; P(x) \mapsto \int_0^\infty e^{-s} P(x+s) dx.$$

Set $\boldsymbol{R} = \boldsymbol{T} - 1$. As explained in [1, II.3.B], the operators $\boldsymbol{T}$ and $\boldsymbol{R}$ are intimately related to the Laguerre polynomials. We have $\boldsymbol{R} = D\boldsymbol{T} = \boldsymbol{T}D$ and[3]

$$\sigma_{\boldsymbol{T}}(t) = \frac{1}{1-t}\sigma_{\boldsymbol{R}}(t) = \frac{1}{1-t} - 1 = \frac{t}{1-t}.$$

If $P \in \mathbb{R}[x]$ is a polynomial of degree m then

$$\boldsymbol{T}^k P(x)_{x=0} = (1 + D + \cdots + D^m)P(x)_{x=0}$$

$$= \int_{\mathbb{R}^k_{\geq 0}} e^{-(s_1+s_2+\cdots+s_k)} P(s_1 + \cdots + s_k) ds_1 \cdots ds_k.$$

For $t \geq 0$ we denote by $\Delta_k(t)$ the $(k-1)$ simplex

$$\Delta_{k-1}(t) := \big\{ (s_1, \dots, s_k) \in \mathbb{R}^k_{\geq 0};\;\; s_1 + \cdots + s_k = t \big\}.,$$

and by $dV_{k-1}(t)$ the Euclidean volume element on $\Delta_{k-1}(t)$. Integrating along the fibers of the function $f : \mathbb{R}^k_{\geq 0} \to [0, \infty)$, $f(s_1, \dots, s_k) = s_1 + \cdots + s_k$ we deduce

$$\int_{\mathbb{R}^k_{\geq 0}} e^{-(s_1+s_2+\cdots+s_k)} P(s_1 + \cdots + s_k) ds_1 \cdots ds_k = \int_0^\infty \left(\int_{\Delta_{k-1}(t)} \frac{1}{|\nabla f|} dV_{k-1}(t) \right) e^{-t} P(t) dt$$

$$= \frac{v_{k-1}}{\sqrt{k}} \int_0^\infty e^{-s} s^{k-1} P(s) ds,$$

where v_{k-1} is the $(k-1)$-dimensional volume of the $(k-1)$-simplex $\Delta_{k-1} = \Delta_{k-1}(t)_{t=1}$.

To compute the volume v_{k-1} we view Δ_k is a regular k-simplex with distinguished base Δ_k, and distinguished vertex $(0, \dots, 0, 1) \in \mathbb{R}^{k+1}$. The distance d_k from the vertex to the base is the distance from the vertex to the center of the base. We have

$$d_k^2 = 1 + \frac{1}{k},\;\; d_k = \sqrt{\frac{k+1}{k}},\;\; v_k = \frac{1}{k} d_k v_{k-1} = \left(\frac{k+1}{k^3}\right)^{1/2} v_{k-1}.$$

Since $v_0 = 1$ we deduce

$$v_k = \frac{(k+1)^{1/2}}{k!},\;\; \boldsymbol{T}^k P(x)_{x=0} = \frac{1}{(k-1)!} \int_0^\infty e^{-s} s^{k-1} P(s) ds,$$

and

$$\boldsymbol{R}^k P(x)_{x=0} = \frac{1}{(k-1)!} \int_0^\infty e^{-s} s^{k-1} P^{(k)}(s) ds.$$

[3] We can write formally $\boldsymbol{T} = \int_0^\infty e^{-s} \boldsymbol{U}^s ds = \int_0^\infty e^{-s(1-D)} ds = (1+D)^{-1}$, so that $\sigma_{\boldsymbol{T}}(t) = \frac{1}{1-t}$.

Using Theorem 1.1 and Corollary 2.4 with the C-convenient series $f(t) = (1+t)^{-1}$ we deduce

$$
\begin{aligned}
{}^{C}\sum_{n\geq 0}(-1)^n \boldsymbol{T}^n P(x)_{x=0} &= {}^{C}\sum_{n\geq 0}(-1)^n \frac{1}{(n-1)!}\int_0^{\infty} e^{-s}s^{n-1}P(s)ds \\
&= \int_0^{\infty}\left(\sum_{k=0}^{\deg P}\frac{(-1)^k}{2^{k+1}(k-1)!}s^{k-1}P^{(k)}(s)\right)ds.
\end{aligned}
$$

If we let $P(s) = s^m$ we deduce

$$
\int_0^{\infty} e^{-s}s^{n-1}P(s)ds = (m+n-1)!,\quad \int_0^{\infty} e^{-s}s^{k-1}P^{(k)}(s)ds = [m]_k(m-1)! = [m-1]_{k-1}m!,
$$

and

$$
{}^{C}\sum_{n\geq 0}(-1)^n\binom{m+n-1}{m} = \sum_{k=0}^{m}\frac{(-1)^k}{2^{k+1}}\binom{m-1}{k-1}. \tag{2.9}
$$

Let us point out that (2.9) can be obtained from (2.8) using the shift-invariance of the Cesàro regularization method. □

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