Regularization of Certain Divergent Series of Polynomials

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Abstract. We investigate the generalized convergence and sums of series of the form \( \sum_{n \geq 0} a_n T^n P(x) \), where \( P \in \mathbb{R}[x] \), \( a_n \in \mathbb{R} \), \( \forall n \geq 0 \), and \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) is a linear operator that commutes with the differentiation \( \frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x] \).

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1 The main result

We consider series of the form

\[ \sum_{n \geq 0} a_n T^n P(x), \quad (\dagger) \]

where \( P \in \mathbb{R}[x] \), and \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) is a linear operator such that

\[ TD = DT, \quad (*) \]

where \( D \) is the differentiation operator \( D = \frac{d}{dx} \). The condition \((*)\) is equivalent with the translation invariance of \( T \), i.e.,

\[ TU^h = U^h T, \quad \forall h \in \mathbb{R}, \quad (I) \]

where \( U^h : \mathbb{R}[x] \to \mathbb{R}[x] \) is the translation operator

\[ \mathbb{R}[x] \ni p(x) \mapsto p(x+h) \in \mathbb{R}[x]. \]

For simplicity we set \( U := U^1 \). Clearly \( U^h \in \mathcal{O} \) so a special case of the series \((\dagger)\) is the series

\[ \sum_{n \geq 0} a_n U^n h P(x) = \sum_{n \geq 0} a_n P(x + nh), \quad h \in \mathbb{R}, \quad (\dagger_h) \]

which is typically divergent.

We denote by \( \mathcal{O} \) the \( \mathbb{R} \)-algebra of translation invariant operators. We have a natural map

\[ \mathcal{Q} : \mathbb{R}[t] \to \mathcal{O}, \quad \mathbb{R}[t] \ni \sum_{n \geq 0} \frac{c_n}{n!} t^n \mapsto \sum_{n \geq 0} \frac{c_n}{n!} D^n. \]
It is known (see [1, Prop. 3.47]) that this map is an isomorphism of rings. We denote by $\sigma$ the inverse of $Q$

\[ \sigma : O \rightarrow \mathbb{R}[\llbracket t \rrbracket], \quad O \ni T \mapsto \sigma_T \in \mathbb{R}[\llbracket t \rrbracket]. \]

For $T \in O$ we will refer to the formal power series $\sigma_T$ as the symbol of the operator $T$. More explicitly

\[ \sigma_T(t) = \sum_{n \geq 0} \frac{c_n(T)}{n!} t^n, \quad c_n(T) = (Tx^n)_{x=0} \in \mathbb{R}. \]

We denote by $\mathbb{N}$ the set of nonnegative integers, and by $\text{Seq}$ the vector space of real sequences, i.e., maps $a : \mathbb{N} \rightarrow \mathbb{R}$. Let $\text{Seq}^c$ the vector subspace of $\text{Seq}$ consisting of all convergent sequences.

A generalized notion of convergence\(^1\) or regularization method is a pair $\mu = (\mu \text{lim, Seq}_\mu)$, where

- $\text{Seq}_\mu$ is a vector subspace of $\text{Seq}$ containing $\text{Seq}^c$ and,
- $\mu \text{lim}$ is a linear map

\[ \mu \text{lim} : \text{Seq}_\mu \rightarrow \mathbb{R}, \quad \text{Seq}_\mu \ni a \mapsto \mu \text{lim} a(n) \in \mathbb{R} \]

such that for any $a \in \text{Seq}^c$ we have

\[ \mu \text{lim} a = \lim_{n \rightarrow \infty} a(n). \]

The sequences in $\text{Seq}_\mu$ are called $\mu$-convergent and $\mu \text{lim}$ is called the $\mu$-limit. To any sequence $a \in \text{Seq}$ we associate the sequence $S[a]$ of partial sums

\[ S[a](n) = \sigma^n_{k=0} a(k). \] (1.1)

A series $\sum_{n \geq 0} a(n)$ is said to by $\mu$-convergent if the sequence $S[a]$ is $\mu$-convergent. We set

\[ \mu \sum_{n \geq 0} a(n) := \mu \text{lim}_n S[a](n). \]

We say that $\mu \sum_{n \geq 0} a(n)$ is the $\mu$-sum of the series. The regularization method is said to be shift invariant if it satisfies the additional condition

\[ \mu \sum_{n \geq 0} a(n) = a(0) + \mu \sum_{n \geq 1} a(n). \] (1.2)

We refer to the classic [3] for a large collection of regularization methods.

For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ we set

\[ [x]_k := \begin{cases} \prod_{i=0}^{k-1} (x - i), & k \geq 1, \\ 1, & k = 0, \end{cases} \quad \left( \begin{array}{c} x \\ k \end{array} \right) := \frac{[x]_k}{k!}. \]

We can now state the main result of this paper.

\(^1\)Hardy refers to such a notion of convergence as convergence in some ‘Pickwickian’ sense.
Theorem 1.1. Let $\mu$ be a regularization method, $T \in O$ and $f(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{R}[[t]]$. Set $c := c_0(T) = T^1$. Suppose that $f$ is $\mu$-regular at $t = c$, i.e.,

for every $k \in \mathbb{N}$ the series $\sum_{n \geq 0} a_n[n]_k c^{n-k}$ is $\mu$-convergent. \((\mu)\)

We denote by $f^{(k)}(c)_\mu$ its $\mu$-sum

$$f^{(k)}(c)_\mu := \mu \sum_{n \geq 0} a_n[n]_k c^{n-k}.$$  

Then for every $P \in \mathbb{R}[x]$ the series $\sum_{n \geq 0} a_n(T^n P)(x)$ is $\mu$-convergent and its $\mu$-sum is

$$\mu \sum_{n \geq 0} a_n(T^n P)(x) = f(T)_\mu P(x),$$

where $f(T)_\mu \in O$ is the operator

$$f(T)_\mu := \sum_{n \geq 0} f^{(k)}(c)_\mu k!(T - c)^k.$$

(1.3)

Proof. Set $R := T - c$ and let $P \in \mathbb{R}[x]$. Then

$$R = \sum_{n \geq 1} \frac{c_n(T)}{n!} D^n$$

so that

$$R^n P = 0, \ \forall n > \deg P.$$  \(1.4\)

In particular this shows that $f(T)_\mu$ is well defined. We have

$$a_n T^n P = a_n (c + R)^n P = a_n \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) c^{n-k} R^k P = \sum_{k=0}^{\deg P} \left( \begin{array}{c} n \\ k \end{array} \right) c^{n-k} R^k P.$$  

At the last step we used (1.4) and the fact that

$$\left( \begin{array}{c} n \\ k \end{array} \right) = 0, \ \text{if } k > n.$$  

This shows that the formal series $\sum_{n \geq 0} a_n(T^n P)(x)$ can be written as a finite linear combination of formal series

$$\sum_{n \geq 0} a_n(T^n P)(x) = \sum_{k=0}^{\deg P} \frac{R^k P(x)}{k!} \left( \sum_{n \geq 0} a_n[n]_k c^{n-k} \right).$$

From the linearity of the $\mu$-summation operator we deduce

$$\mu \sum_{n \geq 0} a_n(T^n P)(x) = \sum_{k=0}^{\deg P} \frac{R^k P(x)}{k!} \left( \mu \sum_{n \geq 0} a_n[n]_k c^{n-k} \right).$$
\[ P(x) = f(T)P(x) \]

\[ \deg P \sum_{k=0}^{\deg P} \frac{f^{(k)}(c)}{k!} R^k \]

\[ \mu_k P(x) = f(T)P(x) \]

\[ \Box \]

2 Some applications

To describe some consequences of Theorem 1.1 we need to first describe some classical facts about regularization methods.

For any sequence \( a \in \text{Seq} \) we denote by \( G_a(t) \in \mathbb{R}[[t]] \) its generating series. We regard the partial sum construction \( S \) in (1.1) as a linear operator \( S : \text{Seq} \to \text{Seq} \). Observe that

\[ G_{S[a]}(t) = \frac{1}{1-t} G_a(t). \]

We say that a regularization method \( \mu_1 = (\mu_1 \lim, \text{Seq}_{\mu_1}) \) is stronger than the regularization method \( \mu_0 = (\mu_0 \lim, \text{Seq}_{\mu_0}) \), and we write this \( \mu_0 \preceq \mu_1 \), if

\[ \text{Seq}_{\mu_0} \subset \text{Seq}_{\mu_1} \quad \text{and} \quad \mu_1 \lim a(n) = \mu_0 \lim a(n), \quad \forall a \in \text{Seq}_{\mu_0}. \]

The Abel regularization method is defined as follows. We say that a sequence \( a \) is \( A \) convergent if

- the radius of convergence of the series \( \sum_{n \geq 0} a_n t^n \) is at least 1 and
- the function \( t \mapsto (1 - t) \sum_{n \geq 0} a_n t^n \) has a finite limit as \( t \to 1^- \).

Hence

\[ A \lim a(n) = \lim_{t \to 1^-} (1 - t) \sum_{n \geq 0} a_n t^n, \]

and \( \text{Seq}_A \) consists of sequence for which the above limit exists and it is finite. Using (2) we deduce that a series \( \sum_{n \geq 0} a(n) \) is \( A \)-convergent if and only if the limit

\[ \lim_{t \to 1^-} \sum_{n \geq 0} a_n t^n \]

exists and it is finite. We have the following immediate result.

**Proposition 2.1.** Suppose that \( f(z) \) is a holomorphic function defined in an open neighborhood of the set \( \{1\} \cup \{|z|\} \subset \mathbb{C} \). If \( \sum_{n \geq 0} \alpha_n z^n \) is the Taylor series expansion of \( f \) at \( z = 0 \) then the corresponding formal power series \( [f] = \sum_{n \geq 0} \alpha_n t^n \) is \( A \)-regular at \( t = 1 \),

\[ [f]^{(k)}(1)_A = f^k(1), \]

and the series

\[ [f](r)_A = \sum_k \frac{[f]^{(k)}(1)_A}{k!} r^k \]

coincides with the Taylor expansion of \( f \) at \( z = 1 \), and it converges to \( f(1+r) \).

\[ ^2 \text{This was apparently known and used by Euler.} \]
Corollary 2.2. Suppose that $f(z)$ is a holomorphic function defined in an open neighborhood of the set \{1\} $\cup$ \{|z|\} $\subset$ $\mathbb{C}$ and $\sum_{n \geq 0} a_n z^n$ is the Taylor series expansion of $f$ at $z = 0$. Then for every $T$ in $\mathfrak{O}$ such that $c_0(T) = 1$, any $P \in \mathbb{R}[x]$, and any $x \in \mathbb{R}$ we have

$$A \sum_{n} a_n T^n P(x) = \sum_{k \geq 0} \frac{f^k(1)}{k!} (T - 1)^k P(x).$$

Let $k \in \mathbb{N}$. A sequence $a \in \text{Seq}$ is said to be $C_k$-convergent (or Cesàro convergent of order $k$) if the limit

$$\lim_{n \to \infty} \frac{S^k[a](n)}{\binom{n+k}{k}}$$

exists and it is finite. We denote this limit by $C_k \lim a(n)$. A series $\sum_{n \geq 0} a(n)$ is said to be $C_k$-convergent if the sequence of partial sums $S[a]$ is $C_k$ convergent. Thus the $C_k$-sum of this series is

$$C_k \sum_{n \geq 0} a(n) = \lim_{n \to \infty} \frac{S^{k+1}[a](n)}{\binom{n+k}{k}}.$$

More explicitly, we have (see [3, Eq.(5.4.5)])

$$C_k \sum_{n \geq 0} a(n) = \lim_{n \to \infty} \frac{n+1}{\binom{n+k}{k}} \left( \sum_{\nu=0}^{n} \frac{\binom{\nu+k}{k}}{\binom{n+k}{k}} a(n-\nu) \right)$$

Hence

$$C_k \sum_{n \geq 0} a(n) \iff S^{k+1}[a](n) \sim A \binom{n+k}{k} \sim A \frac{n^k}{k!},$$

where

$$a \sim b \iff \lim_{n \to \infty} \frac{a(n)}{b(n)} = 1,$$

if $a(n), b(n) \neq 0$, for $n \gg 0$.

The $C_0$ convergence is equivalent with the classical convergence and it is known (see [3, Thm. 43, 55]) that

$$C_k \prec C_{k'} \prec A, \forall k < k'.$$

Given this fact, we define a sequence to be $C$-convergent (Cesàro convergent) if it is $C_k$-convergent for some $k \in \mathbb{N}$. Note that $C \prec A$. Both the $C$ and $A$ methods are shift invariant, i.e., they satisfy the condition (1.2).

We want to comment a bit about possible methods of establishing $C$-convergence. To formulate a general strategy we need to introduce a classical notation. More precisely, if $f(t) = \sum_{n \geq 0} a_n t^n$ is a formal power series we let $[t^n] f(t)$ denote the coefficient of $t^n$ in this power series, i.e., $[t^n] f(t) = a_n$.

Let $f(t) = \sum_{n \geq 0} a_n t^n$. Then the series $\sum_{n \geq 0} a_n t^n$ $C$-converges to $A$ if and only if there exists a nonnegative real number $\alpha$ such that

$$[t^{\alpha}] \left( (1-t)^{-(\alpha+1)} f(t) \right) \sim \frac{n^\alpha}{\Gamma(\alpha + 1)},$$
where \(\Gamma\) is Euler's Gamma function. For a proof we refer to [3, Thm. 43].

**Definition 2.3.** We say that a power series \(f(t) = \sum_{n \geq 0} a_n t^n\) is Cesàro convenient (or \(C\)-convenient) at 1 if the following hold.

(i) The radius of convergences of the series is \(\geq 1\)

(ii) The function \(f\) is regular at \(z = 1\) and has finitely many singularities \(\zeta_1, \ldots, \zeta_\nu \neq 1\) on the unit circle \(\{|z| = 1\}\).

(iii) There exist \(\varepsilon > 0\) and \(\theta \in (0, \frac{\pi}{2})\) such that \(f\) admits a continuation to the dimpled disk \(\Delta_{\varepsilon, \theta} := \{z \in \mathbb{C}; |z| < 1 + \varepsilon, \arg\left(\frac{z}{\zeta_j} - 1\right) > \theta, \forall j = 1, \ldots, \nu\}\).

(iv) For every singular point \(\zeta_j\) there exists a positive integer \(m_j\) such that
\[f(z) = O\left((z - \zeta_j)^{-m_j}\right)\] as \(z \to \zeta_j, z \in \Delta\).

\(\square\)

The results in [2, Chap. VI] implies that the collection \(\mathcal{R}_C\) of \(C\)-convenient power series is a ring satisfying
\[f \in \mathcal{R}_C \iff \frac{df}{dt} \in \mathcal{R}_C.\]

Invoking [2, Thm VI.5] we deduce the following useful consequence.

**Corollary 2.4.** Let \(f \in \mathbb{R}[[t]]\) be a power series \(C\)-convenient at 1. Then \(f\) is \(C\)-regular at 1 and
\[f^k(1)_C = f^{(k)}(1)_A = f^{(k)}(1).\]

\(\square\)

Using [2, VII.7] we obtain the following useful result.

**Corollary 2.5.** (a) The power series
\[(1 + t)^{-m} = \sum_{n \geq 0} \binom{n + m - 1}{n} (-1)^n, \quad m \geq 1, \quad \log(1 + t) = \sum_{n \geq 1} (-1)^{n+1} \frac{t^n}{n}\]
are \(C\)-regular at 1.

(b) If \(f(z)\) is an algebraic function defined on the unit disk \(|z| < 1\) and regular at \(z = 1\) then the Taylor series of \(f\) at \(z = 0\) is \(C\)-regular at 1.

Recall that the Cauchy product of two sequences \(a, b \in \mathbf{Seq}\) is the sequence \(a \ast b\),
\[a \ast b(n) = \sum_{i=0}^{n} a(n - i)b(i), \quad \forall n \in \mathbb{N}.\]
A regularization method is said to be *multiplicative* if
\[ \sum_{n \geq 0} a(n) = \left( \sum_{n \geq 0} a(n) \right) \left( \sum_{n \geq 0} b(n) \right), \]
for any \( \mu \)-convergent series \( \sum_{n \geq 0} a(n) \) and \( \sum_{n \geq 0} b(n) \). The results of [3, Chap.X] show that the \( C \) and \( A \) methods are multiplicative.

For any regularization method \( \mu \) and \( c \in \mathbb{R} \) we denote by \( \mathbb{R}[[t]]_\mu \) the set of series that are \( \mu \)-regular at \( t = 1 \).

**Proposition 2.6.** Let \( \mu \) be a multiplicative regularization method. Then \( \mathbb{R}[[t]]_\mu \) is a commutative ring with one and we have the product rule
\[ (f \cdot g)(n)(1)_\mu = \sum_{k=0}^{n} \binom{n}{k} f(k)(1)_\mu \cdot g(n-k)(1)_\mu. \]
Moreover, if \( T \in \mathcal{O} \) is such that \( c_0(T) = 1 \) then the map
\[ \mathbb{R}[[t]]_\mu \ni f \mapsto f(T)_\mu \in \mathcal{O} \]
is a ring morphism.

**Proof.** The product formula follows from the iterated application of the equalities
\[ D_t(fg) = (D_tf)g + f(D_tg), \quad (fg)(1)_\mu = f(1)_\mu \cdot g(1)_\mu, \quad f'(1)_\mu = (D_tf)(1)_\mu, \]
where \( D_t : \mathbb{R}[[t]] \to \mathbb{R}[[t]] \) is the formal differentiation operator \( \frac{d}{dt} \). The last statement is an immediate application of the above product rule. \( \square \)

**Remark 2.7.** The inclusion \( \mathbb{R}[[t]]_C \subset \mathbb{R}[[t]]_A \) is strict. For example, the power series
\[ f(z) = e^{1/(1+z)} \]
satisfies the assumption of Proposition 2.1 so that the associated formal power series \([f]\) is \( A \)-regular at 1. On the other hand, the arguments in [3, §5.12] show that \([f]\) is not \( C \)-regular at 1. \( \square \)

Consider the translation operator \( U^h \in \mathcal{O} \). From Taylor’s formula
\[ p(x + h) = \sum_{n \geq 0} \frac{h^n}{n!} D^n p(x) \]
we deduce that
\[ \sigma_{U^h}(t) = e^{th}. \]
Set \( \Delta_h := U^h - 1 \). Using Corollary 2.5 and Theorem 1.1 we deduce the following result.
Corollary 2.8. For any $P \in \mathbb{R}[x]$ we have

$$C \sum_{n \geq 0} (-1)^n P(x + nh) = \frac{1}{2} \left( \sum_{n \geq 0} \frac{(-1)^n}{2^n} \Delta_h^n \right) P(x).$$

(2.1)

Observe that

$$\left( 1 + \frac{1}{2} \Delta_h \right) \left( \sum_{n \geq 0} \frac{(-1)^n}{2^n} \Delta_h^n \right) = 1$$

so that $\frac{1}{2} \sum_{n \geq 0} \frac{(-1)^n}{2^n} \Delta_h^n$ is the inverse of the operator $2 + \Delta_h$. We thus have

$$C \sum_{n \geq 0} (-1)^n P(x + nh) = (2 + \Delta_h)^{-1} P(x) = (1 + U^h)^{-1} P(x).$$

(2.2)

Remark 2.9. Here is a heuristic explanation of the equality (2.2) assuming the Cesàro convergence of the series $\sum_{n \geq 0} (-1)^n P(x + nh)$. Denote by $S(x)$ the Cesàro sum of this series. Then

$$S(x + h) = C \sum_{n \geq 0} (-1)^n P(x + (n + 1)h)$$

$$\overset{(1,2)}{=} - C \sum_{n \geq 0} (-1)^n P(x + h) + P(x) = -S(x) + P(x).$$

Hence

$$S(x + h) + S(x) = P(x), \; \forall x \in \mathbb{R}.$$  

If we knew that $S(x)$ is a polynomial we would then deduce

$$S(x) = (1 + U^h)^{-1} P(x).$$  

\[ \square \]

The inverse of $1 + U^h$ can be explicitly expressed using Euler numbers and polynomials, [4, Eq. (14), p.134]. The Euler numbers $E_k$ are defined by the Taylor expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{k \geq 0} \frac{E_k}{k!} t^k.$$  

Since $\cosh t$ is an even function we deduce that $E_k = 0$ for odd $k$. They satisfy the recurrence relation

$$E_n + \left( \frac{n}{2} \right) E_{n-2} + \left( \frac{n}{4} \right) E_{n-4} + \cdots = 0, \; n \geq 2.$$  

(2.3)

Here are the first few Euler numbers.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_n$</td>
<td>1</td>
<td>-1</td>
<td>5</td>
<td>-61</td>
<td>1,385</td>
<td>-50,521</td>
<td>2,702,765</td>
<td>-199,360,981</td>
<td>19,391,512,145</td>
</tr>
</tbody>
</table>
Then
\[
\frac{1}{1 + U^h} = \frac{U^{-\frac{h}{2}}}{U^{\frac{h}{2}} + U^{-\frac{h}{2}}} = \frac{U^{-\frac{h}{2}}}{e^{\frac{h}{2}} + e^{-\frac{h}{2}}} = \frac{1}{2} U^{-\frac{h}{2}} \frac{1}{\cosh \frac{h}{2}} = \frac{1}{2} U^{-\frac{h}{2}} \sum_{k \geq 0} \frac{E_k h^k}{2^k k!} D^k.
\]

Hence
\[
C \sum_{n \geq 0} (-1)^n P(x + nh) = \frac{1}{2} \sum_{k \geq 0} \frac{E_k h^k}{2^k k!} p^{(k)} \left( x - \frac{h}{2} \right).
\]  \quad (2.4)

When \( P(x) = x^m, h = 1 \), we have
\[
C \sum_{n \geq 0} (-1)^n (x + n)^m = \frac{1}{2} \sum_{k \geq 0} \binom{m}{k} \frac{E_k}{2^k} \left( x - \frac{1}{2} \right)^{m-k}.
\]  \quad (2.5)

Setting \( x = 0 \) and using the equality \( E_{2j+1} = 0, \forall j \) we conclude that
\[
C \sum_{n \geq 0} (-1)^n n^m = \frac{1}{2m+1} \sum_{k \geq 0} (-1)^{m-k} \binom{m}{k} \frac{E_k}{2^k} (m-k) = \frac{(-1)^m}{2m+1} \sum_{k \geq 0} E_{2k} \binom{m}{2k}.
\]  \quad (2.6)

Using (2.3) we deduce that when \( m \) is even, \( m = 2m', m' > 0 \) we have
\[
C \sum_{n \geq 0} (-1)^n n^{2m'} = 0.
\]  \quad (2.7)

For example
\[
1 - 1 + 1 - 1 + \cdots \overset{\text{\( \dagger_0 \)}}{=} \frac{1}{2},
\]
\[
-1 + 2 - 3 + 4 - \cdots \overset{\text{\( \dagger_1 \)}}{=} -\frac{1}{4},
\]
\[
-1 + 2^3 - 3^3 + 4^3 - \cdots \overset{\text{\( \dagger_3 \)}}{=} \frac{1}{8},
\]
\[
-1^5 + 2^5 - 3^5 + 4^5 - \cdots \overset{\text{\( \dagger_5 \)}}{=} -\frac{1}{4}.
\]

When \( P(x) = \binom{x}{m}, x = 0, h = 1 \) then it is more convenient to use (2.1) because
\[
\Delta \left( \binom{x}{k} \right) = \binom{x}{k-1}, \forall k, x.
\]

We deduce
\[
C \sum_{n \geq 0} (-1)^n \binom{n}{m} = \frac{1}{2} \sum_{k=1}^{m} \frac{(-1)^k}{2^k} \binom{0}{m-k} = \frac{(-1)^m}{2m+1}.
\]  \quad (2.8)
Example 2.10. Consider the translation invariant operator

\[ T : \mathbb{R}[x] \to \mathbb{R}[x], \quad P(x) \mapsto \int_0^\infty e^{-s}P(x + s)dx. \]

Set \( R = T - 1 \). As explained in [1, II.3.B], the operators \( T \) and \( R \) are intimately related to the Laguerre polynomials. We have \( R = DT = TD \) and

\[ \sigma_T(t) = \frac{1}{1-t} \sigma_R(t) = \frac{1}{1-t} - 1 = \frac{t}{1-t}. \]

If \( P \in \mathbb{R}[x] \) is a polynomial of degree \( m \) then

\[ T^k P(x)_{x=0} = (1 + D + \cdots + D^m)P(x)_{x=0} = \int_{\mathbb{R}_0^k} e^{-(s_1+s_2+\cdots+s_k)}P(s_1 + \cdots + s_k)ds_1 \cdots ds_k. \]

For \( t \geq 0 \) we denote by \( \Delta_k(t) \) the \((k-1)\) simplex

\[ \Delta_{k-1}(t) := \{(s_1, \ldots, s_k) \in \mathbb{R}_{\geq 0}^k; \ s_1 + \cdots + s_k = t \}, \]

and by \( dV_{k-1}(t) \) the Euclidean volume element on \( \Delta_{k-1}(t) \). Integrating along the fibers of the function \( f : \mathbb{R}_{\geq 0}^k \to [0, \infty), \ f(s_1, \ldots, s_k) = s_1 + \cdots + s_k \) we deduce

\[ \int_{\mathbb{R}_0^k} e^{-(s_1+s_2+\cdots+s_k)}P(s_1 + \cdots + s_k)ds_1 \cdots ds_k = \int_0^\infty \left( \int_{\Delta_{k-1}(t)} \frac{1}{|\nabla f|}dV_{k-1}(t) \right) e^{-t}P(t)dt \]

\[ = \frac{v_{k-1}}{\sqrt{k}} \int_0^\infty e^{-s} s^{k-1}P(s)ds, \]

where \( v_{k-1} \) is the \((k-1)\)-dimensional volume of the \((k-1)\)-simplex \( \Delta_{k-1} = \Delta_{k-1}(1) \).

To compute the volume \( v_{k-1} \) we view \( \Delta_k \) is a regular \( k \)-simplex with distinguished base \( \Delta_k \), and distinguished vertex \((0, \ldots, 0, 1) \in \mathbb{R}^{k+1} \). The distance \( d_k \) from the vertex to the base is the distance from the vertex to the center of the base. We have

\[ d_k^2 = 1 + \frac{1}{k^2}, \quad d_k = \sqrt{k+1}, \quad v_k = \frac{1}{k} d_k v_{k-1} = \left( \frac{k+1}{k^3} \right)^{1/2} v_{k-1}. \]

Since \( v_0 = 1 \) we deduce

\[ v_k = \frac{(k+1)^{1/2}}{k!}, \quad T^k P(x)_{x=0} = \frac{1}{(k-1)!} \int_0^\infty e^{-s} s^{k-1}P(s)ds, \]

and

\[ R^k P(x)_{x=0} = \frac{1}{(k-1)!} \int_0^\infty e^{-s} s^{k-1}P^{(k)}(s)ds. \]

\[^3\text{We can write formally } T = \int_0^\infty e^{-s}U^sds = \int_0^\infty e^{-s(1-D)}ds = (1 + D)^{-1}, \text{ so that } \sigma_T(t) = \frac{1}{1-t}. \]
Using Theorem 1.1 and Corollary 2.4 with the C-convenient series \( f(t) = (1+t)^{-1} \) we deduce
\[
C \sum_{n \geq 0} (-1)^n T^n P(x) = C \sum_{n \geq 0} (-1)^n \frac{1}{(n-1)!} \int_0^\infty e^{-s} s^{n-1} P(s) ds
\]
\[
= \int_0^\infty \left( \sum_{k=0}^{\deg P} \frac{(-1)^k}{2^{k+1} (k-1)!} s^{k-1} P^{(k)}(s) \right) ds.
\]

If we let \( P(s) = s^m \) we deduce
\[
\int_0^\infty e^{-s} s^{n-1} P(s) ds = (m+n-1)!, \quad \int_0^\infty e^{-s} s^{k-1} P^{(k)}(s) ds = [m]_k (m-1)! = [m-1]_{k-1} m!;
\]
and
\[
C \sum_{n \geq 0} (-1)^n \binom{m+n-1}{m} = \sum_{k=0}^m \frac{(-1)^k}{2^{k+1}} \binom{m-1}{k-1}.
\]

Let us point out that (2.9) can be obtained from (2.8) using the shift-invariance of the Cesàro regularization method.

\[\square\]

References