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A Game Theoretic Approach for Solving Multiobjective Linear Programming Problems

Irinel DRAGAN

Abstract. The nucleolus is one of the important concepts of solution in the theory of cooperative n-person games with transferable utilities (TU-games). In the set of efficient and individually rational outcomes, the nucleolus is looking for the outcome which is minimizing the unhappiness of the most unhappy coalition, of the second most unhappy coalition, etc. The unhappiness is measured by the values of the excess functions of coalitions relative to the outcome, taken in nonincreasing order. D. Schmeidler (1969) proved that on this set of outcomes, called the imputations, the nucleolus is unique, under some regularity conditions. Kopelowitz (1967) has given a method to compute it, followed by other methods due to Bruynel (1978), Maschler-Peleg-Shapley (1979), Dragan (1981), Potters-Reijnierse-Ansing (1996), etc. The generalized nucleolus, due to M. Justman (1977), is considering as a set of constraints, to replace the constraints of the imputations from game theory, the intersection of a finite number of half spaces, defined by a linear system of inequalities, and is building a criterion of unhappiness, similar to the one from game theory, by means of a group of linear objective functions, replacing the excess functions. In general, the solution is not unique, under some regularity conditions reminding the definition of the nucleoulus.

In the present paper, we state any linear multiobjective problem as the problem of finding the generalized nucleolus, we show how such a problem can be solved, by giving a method similar to the method of finding the nucleolus, where some duality theory is used, and illustrate it by an example. A motivation for the fact of using the generalized nucleolus, even though it may not provide a unique solution, is the fact that it provides a Pareto optimal solution.

Keywords: nucleolus, generalized nucleolus, simplex method, duality theory.

1 The Multiobjective Linear Programming Problems and the Generalized Nucleolus

Consider the multiobjective linear programing (MOLP) problem, defined by a feasible set

$$\Omega = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}, \tag{1.1}$$

and a set of p linear functions to be minimized

$$U = \{u_k(x) : u_k(x) = \alpha_k^T x + \beta_k,\, k = 1, \dots, p\}, \tag{1.2}$$

where A is a $m \times n$ matrix, $b \in \mathbb{R}^m$, $\alpha_k \in \mathbb{R}^n$, and $\beta_k \in \mathbb{R}$, for $k = 1, \dots, p$. It is well known that in general a vector x minimizing one of the functions in U does not minimize the others. Therefore, a concept of solution for a problem like (1) and (2), with $p \geq 2$, should be defined.

In Game Theory, if (N, ν) is a cooperative n - person game with transferable utilities (TU-game), that is N is a finite set, $|N| = n$, and ν is a function defined on the set of subsets of N, with $\nu(\emptyset) = 0$, then any $x \in \mathbb{R}^n$ satisfying

$$\sum_{i \in N} x_i = \nu(N), \quad x_i \geq \nu(\{i\}),\, \forall\, i \in N, \tag{1.3}$$

is an efficient and coalitional rational outcome of the game, called an imputation (see [6]). For each coalition S, the excess of S associated with an imputation x is $e(S, x) = \nu(S) - x(S)$, where as usual $x(S) = \sum_{i \in S} x_i$; for the set of all coalitions we have a set of $p = 2^n - 2$ linear functions

$$E = \{e(S, x) :\, e(S, x) = \nu(S) - x(S),\, S \subset N,\, S \neq \emptyset\}. \tag{1.4}$$

Obviously, the grand coalition which will always have excess zero, due to (1.3), will be missing, and the subset $S = \emptyset$, which is not considered a coalition, is also missing. The larger is the excess of some coalition S, the unhappier is the coalition, because the smaller is the total gain $x(S)$ of the coalition. Therefore, on the set of imputations, we are looking for outcomes which minimize the excesses of the coalitions. We recognize that (1.3) and (1.4) is a multiobjective linear programming problem, in which (1.3) has the form (1.1) and the set E is similar to (1.2). Of course, in (1.3) we get the form (1.1) if we take as new variables the differences $x_i - \nu(\{i\})$, and in (1.4) the vectors α_k are the characteristic vectors of coalitions, to get the functions of the form (1.2).

In [8], D. Schmeidler has introduced the nucleoulus of a TU game as follows: for each imputation x, consider the vector $\theta(x)$ of the excesses taken in a nonincreasing order; then, the nucleoulus of the game is the set $Nu(N, \nu)$ defined by

$$Nu(N, \nu) = \{x' \in I :\, \theta(x') \leq_L \theta(x),\, \forall x \in I\}, \tag{1.5}$$

where $\leq_L$ is the lexicographic ordering of the vector of excesses and I is the set of imputations. A full description of the topic can be found in [6].

In [3], M. Justman has introduced the generalized nucleolus associated with a set of linear constraints (1.1), and a set of linear functions (1.2) to be minimized, as follows: for each x, a feasible solution of (1.1), consider the vector $\theta(x)$ of the values of the linear functions (1.2) taken in a nonincreasing order; then the generalized nucleolus is the set $Nu(\Omega, U)$ defined by

$$Nu(\Omega, U) = \{x' \in \Omega :\, \theta(x') \leq_L \theta(x),\, \forall\, x \in \Omega\}. \tag{1.6}$$

A comparison of the properties of the generalized nucleolus and the nucleolus follows the definition in [3]. The following result from [3] is straight forward:

If Ω is a nonempty compact set and U is a set of continuous functions defined on Ω, then $Nu(\Omega, U) \neq \emptyset$.

Note that these conditions hold in the case of a TU game, so that the nucleolus does exist in that case; moreover, as proved by Schmeidler, another result is helping to prove the uniqueness of the nucleolus of a TU game.

Note also that the interpretation of the generalized nucleolus as the point(s) of the minimal unhappiness of the most unhappy player, the second most unhappy player, etc., is an argument showing that the generalized nucleolus is a "good" concept of solution for the multiobjective problems. Another argument leading to the same conclusion is offered by the application of the theorem to the case of multiobjective problems:

Theorem 1.1 *If Ω is a nonempty compact set, an intersection of half spaces defined in (1.1), and U is a set of linear functions (1.2), defined on Ω, then any point of the generalized nucleolus is a Pareto optimum point of Ω relative to U.*

Proof: Recall that a point $x^\star \in \Omega$ is a Pareto optimum point of Ω, relative to the set of functions U, when there does not exist a point $x \in \Omega$, such that

$$u_k(x) \leq u_k(x^\star), \quad k = 1, \dots, p, \tag{1.7}$$

and

$$u_l(x) < u_l(x^\star), \text{ for some } l \in \{1, \dots, p\}. \tag{1.8}$$

Suppose that $x^\star \in Nu(\Omega, U)$ is not a Pareto optimum point of Ω; then, there is a point $x \in \Omega$ such that (1.7) and (1.8) hold. We can assume, without loss of generality, that $\theta_k(x) = u_k(x)$, $\forall k = 1, \dots, p$, that is the numbering of functions was chosen in such a way that we have

$$u_1(x) \geq u_2(x) \geq \dots \geq u_p(x). \tag{1.9}$$

Consider the p-vector $\eta(x^\star) = (u_k(x^\star))$, whose coordinates are the values of the already numbered functions at the point $x^\star$. We can not say that $\theta_k(x^\star) = u_k(x^\star)$ for some k. According to our hypotheses (1.7) and (1.8), we get $\theta_k(x) \leq \eta_k(x^\star)$, $k = 1, \dots, p$, and $\theta_l(x) < \eta_l(x^\star)$ for some $l \in \{1, \dots, p\}$. As we have

$$\theta_1(x) \leq \eta_1(x^\star) \leq \max u_k(x^\star) = \theta_1(x^\star), \tag{1.10}$$

the inequality $\theta_1(x) \leq \theta_1(x^\star)$ holds. But we have $x^\star \in Nu(\Omega, U)$, and $x \in \Omega$ shows that the inequality $\theta_1(x) \leq \theta_1(x^\star)$ can not be true; hence, $\theta_1(x) = \eta_1(x^\star) = \theta_1(x^\star)$. If we consider the pair $\theta_2(x)$ and $\eta_2(x^\star)$, and we repeat the reasoning, we get $\theta_2(x) = \eta_2(x^\star) = \theta_2(x^\star)$. Now, by induction, we prove that $\theta_k(x) \leq \eta_k(x^\star)$, $k = 1, \dots, p$, are satisfied as equalities. This fact contradicts the hypothesis $\theta_l(x) < \eta_l(x^\star)$ for some index $l \in \{1, \dots, p\}$, the theorem follows. □

The meaning of Theorem 1.1 is that by taking any point in the set $Nu(\Omega, U)$ as a solution of the MOLP problem, we are taking in the set of Pareto optimum points a point with supplementary properties.

2 Finding the generalized nucleolus as a solution for an MOLP problem

The main idea of the algorithm to be pressented below for solving MOLP problems will be that of replacing in each step the given problem by a new MOLP problem in which some of the objective functions are included in the constraints defining the new feasible set, equated to some constants; recall that the same idea was used by Kopelowitz in his algorithm for computing the nucleolus (see [4] and [6]). In this way, the number of the objective functions is reduced in any step of the algorithm, until the objective functions are exhausted. The algorithm keeps the generalized nucleolus of the initial feasible set in the successive feasible sets and one proves that the final feasible set represents just the generalized nucleolus, i.e. the solution of the MOLP problem. Moreover, one point in the generalized nucleolus is also obtained by the algorithm. The method was suggested by [4] and [5]. The results needed to justify the algorithm are presented in this section, while the algorithm will be stated in the next section.

Associated with the MOLP problem (1.1) and (1.2), consider the linear programming problem (P):

$$\min\{t : x \in \Omega,\, u_k(x) \leq t,\, k = 1, \ldots, p\}. \tag{2.1}$$

As the second group of restrictions can be written as $\max u_k(x) \leq t$, it is clear that if $\Omega \neq \emptyset$, the problem (P) has feasible solutions and vice versa. Thus, the case $\Omega = \emptyset$, which is making the MOLP problem senseless, is discovered in solving the problem (P). Otherwise, if Ω is a nonempty compact set, the problem (P) has an optimal solution. The crucial point in the construction of a primal method for solving the MOLP problem is the following:

Lemma 2.1 *If Ω is a nonempty compact set and $t^\star$ is the optimal value of (P), then there is an integer $p^\star$, $(1 \leq p^\star \leq p)$, and a set of indices $k_{i_1}, \ldots, k_{i_{p^\star}}$ in $\{1, \ldots, p\}$, such that for any optimal solution of (P), let say $(x^\star, t^\star)$, we have*

$$u_{k_i}(x^\star) = t^\star, \quad i = 1, \ldots, p^\star. \tag{2.2}$$

Proof: As $t^\star$ is the optimal value of (P), suppose that for each $k \in \{1, \ldots, p\}$, there is an optimal solution $(x^k, t^\star)$ such that $u_k(x^k) < t^\star$, where $x^k \in \Omega$, $k = 1, \ldots, p$, are distinct vectors, or not. We want to show that this can not be true.
Consider the vector

$$x = \frac{1}{p}\sum_{k=1}^{p} x^k, \tag{2.3}$$

which is a point in Ω, because this is a convex set. For any $j \in \{1, \ldots, p\}$, we have

$$u_j(x^k) \leq t^\star, \text{ if } j \neq k, \quad u_k(x^k) < t^\star. \tag{2.4}$$

From (2.3), by the linearity of the objective functions, we get $u_j(x) < t^\star$, for all $j \in \{1, \ldots, p\}$. This says that $(x, \max u_j(x))$ is a feasible solution of (P), which gives for the objective function a lower value than $t^\star$, the optimal value. The contradiction proves the lemma. □

Note that the lemma says that by adding to the constraints of problem (P) the equations $u_{k_i}(x) = t^\star$, for $i = 1, \dots, p^\star$, no optimal solution has been lost. This is exactly what we want to do in order to find the generalized nucleolus, for which the elements $x^\star$ will be further proved to form pairs $(x^\star, t^\star)$ included in the set of optimal solutions of our problem (P).

Obviously, we should explain how the indices of the functions appearing in these equations (2.2) can be found, a fact to be done at the end of this section. Notice that the lemma uses in the proof the linearity of the objective functions, hence the extension to the nonlinear case is really a problem.

Example 2.2 *Consider the MOLP problem*

$$\text{Minimize} \quad u_1(x) = 2x + y, \quad u_2(x) = x - 2y,$$

on the set Ω*, defined by the system of linear inequalities*

$$x - y \leq 1,\ -x + y \leq 2,\ x + y \leq 3,\ x + y \geq 1, \quad x \geq 0,\ y \geq 0.$$

Note that t *is an unconstrained variable. The problem (P) shown in (2.1) is:*
Minimize t*, subject to*

$$x - y \leq 1,\ -x + y \leq 2,\ x + y \leq 3,\ x + y \geq 1, 2x + y \leq t,\ x - 2y \leq t, \quad x \geq 0, y \geq 0.$$

We can compute an optimal solution for this problem: $x^\star = 0, y^\star = 1, t^\star = 1$. One may check its feasibility directly, and the optimality, if we write the dual problem and the complementarity conditions. Moreover, as it will be explained at the end of the section, we shall be able to determine that for any optimal solution of (P), we have $2x + y = 1$; hence, $p^\star = 1$, and this equation should be added to the constraints of problem (P), to form a new MOLP problem.

Note that in the above problem the optimal solution is unique; therefore, there is no need to form a new MOLP problem and solve it. In general, there are several optimal solutions for the problem (P), so that we should form the second MOLP problem and solve it. If the optimal solution is unique, then for this solution we can compute the corresponding vector $\theta(x^\star)$ and in the vector will appear the optimal values of t for the next problems.

Now, suppose that by some method it has been found a set of indices Π, with $|\Pi| = p^\star$, $1 \leq p^\star \leq p$, such that for any optimal solution $(x^\star, t^\star)$ of (P), we have

$$u_k(x^\star) = t^\star, \quad \forall\, k \in \Pi, \tag{2.5}$$

and two cases may arise: either $p^\star = p$, or $p^\star < p$. Obviously, if $p^\star < p$, then we have beside (2.5) the inequalities

$$u_k(x^\star) < t^\star, \forall\, k \notin \Pi. \tag{2.6}$$

Theorem 2.4 discusses the first case, which will be our stopping criterion of the algorithm, while Lemma 2.5 discusses the second case, and leads to the algorithm further justified by Theorem 2.6. The following result on a general property of any element in the generalized nucleolus is valid in both cases, $p^\star = p$, and $p^\star < p$.

Lemma 2.3 *If Ω is a nonempty compact set, $t^\star$ is the optimal value of problem (P), and $p^\star \leq p$ is the integer just defined, then for any element $x^\star \in Nu(\Omega, U)$, we get $(x^\star, t^\star)$ is an optimal solution of problem (P), and we have*

$$\theta_1(x^\star) = \cdots = \theta_{p^\star}(x^\star) = t^\star. \tag{2.7}$$

Proof: Let $(x', t^\star)$ be an optimal solution of (P); then, $x' \in \Omega$ and we have by (1.6) that $\theta(x^\star) \leq_L \theta(x')$. Thus, $\theta_1(x^\star) \leq \theta_1(x')$, and according to the definition of $p^\star$, we get $\theta_1(x') = t^\star$, that is $\theta_1(x^\star) \leq t^\star$. This inequality and the definition of $\theta(x^\star)$, imply $u_k(x^\star) \leq t^\star$, for all $k = 1, \ldots, p$, so that $(x^\star, t^\star)$ is an optimal solution of problem (P). Obviously, the maximal value of the objectives at $x^\star$ is $t^\star$, otherwise $t^\star$ would not be the optimal value. The equalities (2.7) will follow from the definition of $p^\star$. □

Note that by the lemma, no element in the generalized nucleolus is lost by adding (2.5) to the set of constraints. Therefore, in the second case, $p^\star < p$, we should further consider the new MOLP problem of minimizing the objectives $u_k(x)$, $\forall k \notin \Pi$, on the new feasible set

$$\Omega^\star = \{x \in \Omega :\ u_k(x) = t^\star,\ \forall k \in \Pi\}, \tag{2.8}$$

where $t^\star$ is the optimal value of (P) and Π is defined by (2.5) and (2.6).

Now, consider the new set

$$S_{OPT}(P) = \{x \in \Omega :\ (x, t^\star)\,\text{is an optimal solution of}\,(P)\} \subseteq \Omega^\star. \tag{2.9}$$

Lemma 2.3 proves the inclusion and shows that $Nu(\Omega, U) \subseteq S_{OPT}(P)$, hence from (2.9) it follows that $Nu(\Omega, U) \subseteq \Omega^\star$. Recall that the result holds in both cases $p^\star = p$ and $p^\star < p$; but we shall prove now that in the first case we have $S_{OPT} = \Omega^\star$.

Theorem 2.4 *If Ω is a nonempty compact set, and $t^\star$ is the optimal value of (P), and for any optimal solution $(x^\star, t^\star)$ of (P) we have $u_k(x^\star) = t^\star$, for $k = 1, \ldots, p$, then we obtain $Nu(\Omega, U) = \Omega^\star$.*

Proof: As we already remarked, we have $Nu(\Omega, U) \subseteq S_{OPT}(P)$. Let us suppose that there is $x' \in S_{OPT}(P)$, such that $x' \notin Nu(\Omega, U)$. Then, there is $x^\star \in Nu(\Omega, U)$ such that $\theta(x^\star) \leq_L \theta(x')$. But, from the hypothesis we get $\theta_k(x') = t^\star$, $\forall\, k = 1, \ldots, p$, and therefore we get $\theta_k(x^\star) = t^\star$, $\forall\, k = 1, \ldots, p$. Hence, $\theta(x') = \theta(x^\star)$, and the contradiction proves the theorem. □

Theorem 2.4. shows that in the case $p^\star = p$ we have $Nu(\Omega, U) = \Omega^\star$, and the MOLP problem is solved by determining the solution set. Now, one solution is available at the end of the computation needed to solve the problem (P), so that a point in $Nu(\Omega, U)$ is already available. If more solutions are needed, then this set may be further explored. Theorem 2.4 is the stopping criterion for the method, and this is the case described earlier by saying that the objective functions have been exhausted.

It remains to consider the case $p^\star < p$; as we already remarked, by Lemma 2.3 we have $Nu(\Omega, U) \subseteq \Omega^\star$, hence $\Omega^\star \neq \emptyset$. We form the new MOLP problem $(P^\star)$, asking to minimize the functions in $U^\star = \{u_k(x) :\ u_k(x), \forall\, k \notin \Pi\}$, on $\Omega^\star$ shown in (2.8). Moreover, $\Omega^\star$ is a compact set, hence $Nu(\Omega^\star, U^\star) \neq \emptyset$. Consider further the second case:

Lemma 2.5 *If Ω is a nonempty compact set, $t^\star$ is the optimal value of (P), we have $p^\star < p$, so that we built the new set $\Omega^\star$, given by (2.8), and the problem $(P^\star)$ is built by using the objectives $U^\star$, then $x^\star \in Nu(\Omega^\star, U^\star)$ implies that $(x^\star, t^\star)$ is an optimal solution of $(P^\star)$, and the equalities*

$$\theta_1(x^\star) = \cdots = \theta_{p^\star}(x^\star) = t^\star, \tag{2.10}$$

hold.

Proof: Let $(x', t^\star)$ be an optimal solution of (P); then, $x' \in \Omega^\star$ and we have $\theta(x^\star) \leq_L \theta(x')$. Thus, $\theta_1(x^\star) \leq \theta_1(x')$ and according to the definition of $p^\star$, we get $\theta_1(x') = t^\star$, i.e. $\theta_1(x^\star) \leq t^\star$. This inequality and the definition of $\theta(x^\star)$ imply

$$u_k(x^\star) \leq t^\star, \quad k = 1, \ldots, p, \tag{2.11}$$

so that $(x^\star, t^\star)$ is an optimal solution of (P). Obviously, $\max u_k(x^\star) = t^\star$, otherwise $t^\star$ would not be the optimal value. The equalities given in the lemma follow from the definition of $p^\star$. □

Notice that Lemma 2.5 has proved that we have $Nu(\Omega^\star, U^\star) \subseteq S_{OPT}(P) \subseteq \Omega^\star \subseteq \Omega$. It remains to be seen why we should continue to investigate $Nu(\Omega^\star, U^\star)$, by creating a new linear programming problem $(P^\star)$, and how this procedure may end by solving the new problem.

Theorem 2.6 *If Ω is a nonempty compact set, and $t^\star$ is the optimal value of (P), consider the set $\Omega^\star = \{x \in \Omega : u_k(x) = t^\star, \forall\, k \in \Pi\}$, where Π was defined by (2.5) and (2.6), then we have $Nu(\Omega, U) = Nu(\Omega^\star, U^\star)$.*

Proof: We shall be proving the double inclusion, to conclude the equality of the two sets; first, prove $Nu(\Omega, U) \subseteq Nu(\Omega^\star, U^\star)$. Let $x^\star \in Nu(\Omega, U)$; as $Nu(\Omega, U) \subseteq \Omega^\star$, we have $x^\star \in \Omega^\star$. If $x^\star \notin Nu(\Omega^\star, U^\star)$, then there is $x' \in Nu(\Omega^\star, U^\star)$ such that $\theta(x') \leq_L \theta(x^\star)$. From Lemma 2.3 and Theorem 2.4, we have

$$\theta_1(x') = \theta_1(x^\star), \ldots, \theta_{p'}(x') = \theta_{p'}(x^\star); \tag{2.12}$$

hence, there is an index k', $p' \leq k' < p$, so that $\theta_k(x') = \theta_k(x^\star), \forall\, k = 1, \ldots, k'$, and $\theta_{k'+1}(x') < \theta_{k'+1}(x^\star)$. As $x' \in \Omega$, the existence of k' contradicts $x^\star \in Nu(\Omega, U)$. In consequence, $Nu(\Omega, U) \subseteq Nu(\Omega^\star, U^\star)$. Second, we prove the opposite inclusion; let $x' \in Nu(\Omega^\star, U^\star)$ and $x' \notin Nu(\Omega, U)$; as $x' \in \Omega$, we have $\theta(x^\star) \leq_L \theta(x')$ for any $x^\star \in Nu(\Omega, U)$. From Lemma 2.3 and Theorem 2.4, we have

$$\theta_1(x^\star) = \theta_1(x'), \ldots, \theta_{p'}(x^\star) = \theta_{p'}(x'), \tag{2.13}$$

hence, there is an index k', $p^\star \leq k' < p$, so that $\theta_k(x^\star) = \theta_k(x')$, $\forall\, k = 1, \ldots, k'$, and $\theta_{k'+1}(x^\star) < \theta_{k'+1}(x')$. As $Nu(\Omega, U) \subseteq Nu(\Omega^\star, U^\star)$ implies $x^\star \in Nu(\Omega^\star, U^\star)$, the last result contradicts $x' \in Nu(\Omega^\star, U^\star)$. It follows $x' \in Nu(\Omega, U)$, in consequence we shall obtain $Nu(\Omega^\star, U^\star) \subseteq Nu(\Omega, U)$. The equality stated in the theorem holds. □

Now, the above Theorem 2.6 justifies the step of the algorithm in case that $p^\star < p$, i.e. the objective functions have not been yet exhausted. Precisely, for finding the generalized nucleolus of the initial MOLP problem, we can solve a new MOLP problem on the feasible set $\Omega^\star$, defined in (2.8), with respect to the objective functions

$$U^\star = \{u_k(x) : u_k(x) = \alpha_k^T x + \beta_k,\ k \notin \Pi\}. \tag{2.14}$$

It remains to explain how can we find a set of indices Π introduced in Lemma 2.1; some elements of the duality theory of linear programming will be used. Consider the dual problem of (P), taking into account that t is unconstrained and Ω can be written under the form

$$\Omega = \{x \in \mathbb{R}^n : A_i x \leq b_i,\ i = 1, \ldots, m;\ x \geq 0\}, \tag{2.15}$$

where A_i is the row i of A. The problem (P) is:

$$\min\{t : A_i x \leq b_i,\ i = 1, \ldots, m;\ \alpha_k^T x - t \leq -\beta_k,\ k = 1, \ldots, p;\ x \geq 0\}. \tag{2.16}$$

Then, the dual problem (D) is

$$\max\{-\sum_1^m b_i\sigma_i + \sum_1^p \beta_k\tau_k : \sum_1^m A_i\sigma_i + \sum_1^p \alpha_k\tau_k \geq 0;\ \sum_1^p \tau_k = 1;\ \sigma \geq 0,\ \tau \geq 0\}. \tag{2.17}$$

where $\sigma \in \mathbb{R}^m$ and $\tau \in \mathbb{R}^p$ are the vectors of the dual variables. The complementarity conditions are

$$\sigma_i(b_i - A_i x) = 0,\ i = 1, \ldots, m, \quad \tau_k(t - u_k(x)) = 0,\ k = 1, \ldots, p. \tag{2.18}$$

Suppose that the problem (P) has been solved. Then, for any optimal solution $(x^\star, t^\star)$ of (P), and any optimal solution of the dual $(\sigma^\star, \tau^\star)$ of (P), by complementary slackness theorem, the complementarity conditions should be satisfied; among them

$$\tau_k^\star(t^\star - u_k(x^\star)) = 0,\ k = 1, \ldots, p. \tag{2.19}$$

From (2.17), we have $\sum_1^p \tau_k^\star = 1$, hence in $\tau^\star$ there is at least one positive coordinate. If we denote

$$\Pi = \{k : k \in \{1, \ldots, p\},\ \tau_k^\star > 0\}, \tag{2.20}$$

then, for any optimal solution $(x^\star, t^\star)$ of (P), we have

$$u_k(x^\star) = t^\star, \quad \forall\, k \in \Pi. \tag{2.21}$$

If $|\Pi| = p$, the MOLP problem is solved, as explained by Theorem 2.4, if $|\Pi| = p^\star < p$, then the MOLP problem should be replaced by a new one, as explained by Theorem 2.6.

Example 2.7 *Return to Example 2.2, and write the dual problem (2.17) for the problem discussed in Example 2.2:*

$$\textit{Minimize} \quad g = \sigma_1 + 2\sigma_2 + 3\sigma_3 - \sigma_4,$$

subject to

$$\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + 2\tau_1 + \tau_2 \geq 0, \quad -\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4 + \tau_1 - 2\tau_2 \geq 0,$$
$$\tau_1 + \tau_2 = 1, \quad \sigma_1 \geq 0, \quad \sigma_2 \geq 0, \quad \sigma_3 \geq 0, \quad \sigma_4 \geq 0.$$

Among the complementarity conditions we have

$$\tau_1(t - 2x - y) = 0, \quad \tau_2(t - x + 2y) = 0.$$

For the optimal solution which has been found by solving the problem (P), i.e. $x^\star = 0$, $y^\star = 1$, $t^\star = 1$, together with $\tau_1 + \tau_2 = 1$, give $\tau_1^\star = 1$, $\tau_2^\star = 0$. As explained above in (2.20) and (2.21), we obtain $\Pi = \{1\}$, so that for any optimal solution of (P) we should have $2x + y = 1$, equation to be added to the constraints in (P), in order to get the new MOLP problem, to be further solved for getting the generalized nucleolus of the initial problem. This is: minimize the function

$$u_2(x) = x - 2y,$$

on the set $\Omega^\star$ defined by the system of linear inequalities

$$x - y \leq 1,\ -x + y \leq 2,\ x + y \leq 3,\ x + y \geq 1,\ 2x + y = 1,\ x \geq 0,\ y \geq 0.$$

If this problem is solved we get the optimal solution $x^\star = 0$, $y^\star = 1$, $t^\star = -2$. The generalized nucleolus has only one solution $x = 0$, $y = 1$, and the objectives have the values included in the vector $\theta^T = (1, -2)$. Of course, if we know that the solution of (P) is unique, we may compute the values of objectives at that solution and we should not solve the second problem. If there are more objectives and we solved the second MOLP problem without exhausting the objectives, we should go to the third MOLP problem.

3 Algorithm for solving a MOLP problem

The algorithm for solving a MOLP problem, that is for finding a point in the generalized nucleolus of the compact feasible set Ω, relative to the linear functions U, can be stated as follows:

STEP 0: Find out whether $\Omega \neq \emptyset$, or not. In the second case, stop, the MOLP has no solution. Before step s, $s \geq 1$, a feasible set Ω^s and a system of linear functions U^s is available. For $s = 1$, we have $\Omega^1 = \Omega$, $U^1 = U$, $p_1 = |U^1| = p$.

STEP s:

1. Solve the LP problem (P^s):

   $$\text{Minimize}\, t,\ \text{s.t.}\, x \in \Omega^s,\ u_k^s \leq t,\ k = 1, \ldots, p_s.$$

   Let t^s be the optimal value.
2. Find a dual optimal solution (σ^s, τ^s) and determine the set of indices $\Pi^s = \{k : \tau_k^s > 0\}$.
3. Update Ω^s and U^s; $\Omega^s := \{x \in \Omega^s : u_k^s = t^s,\ k \in \Pi^s\}$, then $p_s := p_s - |\Pi^s|$, and $U^s := \{u_k^s : k = 1, \ldots, p_s,\ k \notin \Omega^s\}$;
4. Check whether $p^s = 0$ or $p^s > 0$; in the first case, go to 5, in the second case, take on $s := s + 1$, and go to a new step;
5. Stop the procedure, the solution is the set Ω^s; if only one element of the generalized nucleolus is desired, take the x-part of the optimal solution of the problem (P^s).

The convergence in a finite number of steps is assured by the fact that in each step at least one objective function of the current MOLP problem is incorporated in the restrictions. Hence, the number of steps is at most p. Of course, a dual method of the method given above can be stated, as it has been done in the case of the nucleolus (see [1] and [2]). Finally, let us remark that in the case $p \geq n$, if there are n linearly independent objectives, then the generalized nucleolus consists of exactly one point, because at some step of the algorithm all these n functions will be among the constraints equated with some constants and these equations have only one solution.

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