LIBERTAS MATHEMATICA, vol XXX (2010)

Diagonal Invariance and Comparison Methods

Octavian PASTRAVANU and
Mihaela-Hanako MATCOVSCHI

Abstract. A general framework has been developed to explore the positively (flow) invariant sets, for a large class of time-variant, nonlinear dynamical systems. Thus, we introduce the concept of "diagonal invariance" defined by time-dependent diagonal matrices and for arbitrary Hölder norms. The flow invariance results are formulated as necessary and sufficient conditions for linear systems. The approach to nonlinear systems relies on sufficient conditions that allow formulating a comparison theorem where the comparison system exhibits linear dynamics. We illustrate the applicability of our results for studying the diagonal invariance of a class of nonlinear systems. This framework can be simply adapted to discrete-time systems.

Keywords: continuous-time dynamical systems, invariant sets, comparison methods, matrix measures.

Mathematics Subject Classification (2010): 34A34, 34A30, 93C10, 93C05, 93B25.

1. Introduction

Starting with the period of the eighties, a large body of work has been invested in studying the properties of the invariant sets with respect to the trajectories of dynamical systems. Part of these researches have been summarized by the recent monographs [2], [3], [6], [8], [9], [11], to cite just a few of the contributions brought to the discussed field. Beyond the theoretical importance of the general results, the applicability area is drastically limited to some classes of systems, among which the linear ones polarize the greatest interest. Moreover, most of the results focus on constant (time independent) invariant sets, whereas the case of time-dependent invariant sets has remained almost unexplored, except for a very small number of papers considering the invariant sets generated by time-varying hyper-rectangles [7], [10], [12], [13], [14], [15], [16].

The current paper develops a general framework for the analysis of flow invariant sets defined by arbitrary Hölder norms and exhibiting arbitrary time dependence; this framework is able to incorporate, the results mentioned above as particular cases. The basic concept is the diagonal invariance, introduced in Section 2, for the analysis of which Sections 3 and 4 provide specialized tools. The major contribution of the paper is presented in Section 5 and consists in the formulation of a comparison theorem for diagonal invariance. Section 6 applies the comparison method to a class of nonlinear systems.

2. Concept of Diagonal Invariance

Consider the nonlinear system

$$\dot{x}(t) = f\left(x(t), t\right), \quad x \in \mathbb{R}^n,\ x(t_0) = x_0, \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuously differentiable in $x \in \mathbb{R}^n$, continuous in $t \in \mathbb{R}_+$, and $f(0,t) = 0$, $\forall t \in \mathbb{R}_+$, i.e. $\{0\}$ is an *equilibrium point* (EP) of system (1). The state-space trajectory of (1) initialized in $x(t_0) = x_0$ is denoted by $x(t; t_0, x_0)$.

Let $||\ ||_p : \mathbb{R}^n \to \mathbb{R}_+$ denote the Hölder p-norm in $\mathbb{R}^n$. Given a positive vector function:

$$h(t) = \left[h_1(t) \cdots h_n(t)\right]^T : \mathbb{R}_+ \to \mathbb{R}^n, \quad h_i(t) > 0,\ \forall t \in \mathbb{R}_+,\ i = 1, \ldots, n, \tag{2}$$

where T denotes the transposition, introduce the diagonal matrix

$$H(t) = \operatorname{diag}\left\{h_1(t), \cdots, h_2(t)\right\} \tag{3}$$

and consider the *time-dependent set* (TDS)

$$S_{p,h(t)} = \left\{x \in \mathbb{R}^n |\ ||H^{-1}(t)x||_p \leq 1\right\},\ \ t \geq 0. \tag{4}$$

It is obvious that TDS $S_{p,h(t)}$ (4) is symmetrical and the axes of coordinates corresponding to the variables of the state-space representation (1) play the role of symmetry axes, regardless of the considered Hölder p-norm.

Definition 1. TDS $S_{p,h(t)}$ (4) is *flow (positively) invariant with respect to* (abbreviated as FI w.r.t.) system (1), if any trajectory initiated inside TDS (4) remains inside TDS (4) at any time, i.e.

$$\forall t_0 \in \mathbb{R}_+, \forall x_0 \in S_{p,h(t_0)} \Rightarrow \forall t_1 > t_0, x(t_1; t_0, x_0) \in S_{p,h(t_1)}. \tag{5}$$

For many classes of systems, the existence of an invariant TDS $S_{p,h(t)}$ (4) ensures the existence of a whole set of TDSs that are FI w.r.t. the system trajectories, denoted by $S_{p,\rho h(t)}$ and defined as in (4), but with respect to the vector function $\tilde{h}(t) = \rho h(t)$, $\rho > 0$, instead of $h(t)$. This situation can be regarded as a *system property* formulated as follows:

Definition 2. Given a positive vector function $h(t)$ (2), system (1) is called *locally / globally diagonally-invariant* relative to *Hölder p-norm*, with respect to $h(t)$ (abbreviated as locally / globally $DI_{p,h(t)}$), if the TDS $S_{p,\rho h(t)}$ is FI w.r.t. system (1) for any $\rho \in (0,1]/\rho \in (0,\infty)$.

The nomenclature "diagonally invariant" is motivated by the diagonal form of the matrix $H(t)$ used for defining the TDSs FI w.r.t. system (1).

3. Diagonal Invariance of Nonlinear Systems

Lemma 1. *Given a positive vector function $h(t)$ (2) and an arbitrary Hölder p-norm, if the function*

$$W(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+,\ W(x,t) = \ ||H^{-1}(t)x||_p \tag{6}$$

is nonincreasing along each trajectory of system (1), *then TDS* $S_{p,h(t)}$ (4) *is FI w.r.t. system* (1).

Proof. Assume that TDS (4) is not FI w.r.t. $S_{p,h(t)}$. This means that there exist at least a trajectory $x^*(t)$ initialized inside $S_{p,h(t)}$ and a time instant t^* such that $||H^{-1}(t^*)x^*(t^*)||_p = 1$ and $||H^{-1}(t)x^*(t)||_p > 1$ for $t > t^*$, or, in other words, that $W(t)$ is strictly increasing in a vicinity of t^*. The proof is completed, since we have contradicted the hypothesis of Lemma 1. □

Remark 1. Note that, in general, $W(x,t)$ defined by (6) is not a weak Lyapunov function, since for an arbitrary positive vector function $h(t)$, the function $W(x,t)$ is not necessarily positive definite (see, e.g. [11])

Lemma 2. *System (1) is equivalent to the system:*

$$\dot{x}(t) = A(x(t),t)x(t) \tag{7}$$

where the $n \times n$ *matrix* $A(x,t)$ *is defined by:*

$$A(x,t) = \int_0^1 J(sx,t)ds \tag{8}$$

and $J(x,t) = [\partial f(x,t)/\partial x] \in \mathbb{R}^{n\times n}$ *denotes the Jacobian matrix with respect to* x *of the vector function* $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$.

Proof. Consider the function $\varphi(s,x,t) = f(sx,t)$, as suggested in [5], satisfying $\frac{\partial \varphi}{\partial s} = J(sx,t)x$. The proof consists in expressing $\int_0^1 \frac{\partial \varphi}{\partial s} ds$ by two different forms, namely

$$\int_0^1 \frac{\partial \varphi}{\partial s} ds = f(sx,t)\Big|_0^1 = f(x,t) - f(0,t) = f(x,t)$$

and

$$\int_0^1 \frac{\partial \varphi}{\partial s} ds = \int_0^1 J(sx,t)x ds = \left[\int_0^1 J(sx,t)ds\right] x.$$

□

Given a square matrix $Q \in \mathbb{R}^{n\times n}$, consider its measure [4]

$$\mu_{||\ ||_p}(Q) = \lim_{\theta \downarrow 0} (||I + \theta Q||_p - 1)/\theta, \tag{9}$$

associated with the matrix norm $||Q||_p$ induced by the Hölder p-norm of the vectors in $\mathbb{R}^n$.

Proposition 1. *Consider the set* $\Omega_p \supseteq \bigcup_{t\in\mathbb{R}_+} S_{p,h(t)}$. *Given a positive vector function* $h(t)$ *(2) and an arbitrary Hölder p-norm, if* $\forall t \in \mathbb{R}_+$, $\forall x \in \Omega_p$ *one of the following two inequalities is fulfilled:*

$$(a)\ \mu_{||\ ||_p}\left(H^{-1}(t)A(x,t)H(t) - H^{-1}(t)\dot{H}(t)\right) \le 0, \tag{10a}$$

(b) $\mu_{|| \, ||_p}\left(H^{-1}(t)J(x,t)H(t) - H^{-1}(t)\dot{H}(t)\right) \le 0,$ (10b)

then system (1) is locally $DI_{p,h(t)}$.

Proof. (a) First we prove the invariance of the TDS $S_{p,h(t)}$ (4) w.r.t. system (1), by showing that

$$D_t^+ W(x,t) = \lim_{\theta \downarrow 0} \frac{W(x,t+\theta) - W(x,t)}{\theta} \le 0.$$

Indeed, $\forall t \in \mathbb{R}_+$, $\forall x \in S_{p,h(t)}$ for the unique trajectory of system (1) initialized in $x(t) = x$, we can write

$$H^{-1}(t+\theta)x(t+\theta) = H^{-1}(t)x(t) + \theta \frac{d}{dt}\left(H^{-1}(t)x(t)\right) + \theta O(\theta),$$

with $\lim_{\theta \downarrow 0} ||O(\theta)||_p = 0$ and

$$\frac{d}{dt}\left(H^{-1}(t)x(t)\right) = M(x,t)\left(H^{-1}(t)x(t)\right),$$

where matrix $M(x,t)$ is defined by

$$M(x,t) = H^{-1}(t)A(x,t)H(t) - H^{-1}(t)\dot{H}(t).$$

Hence,

$$W(t+\theta) = ||\left(I + \theta M(x,t)\right)\left(H^{-1}(t)x(t)\right) + \theta O(\theta)||_p \le ||I + \theta M(x,t)||_p W(t) + \theta ||O(\theta)||_p$$

and

$$D_t^+ W(x,t) \le \lim_{\theta \downarrow 0} \frac{||I + \theta M(x,t)||_p - 1}{\theta} W(t) + \lim_{\theta \downarrow 0} O(\theta) = \mu_{|| \, ||_p}\left(M(x,t)\right) W(t).$$

This means inequality (10a) implies $D_t^+ W(x,t) \le 0$, i.e. $W(x,t)$ (8) is nonincreasing and Lemma 1 ensures that $S_{p,h(t)}$ (4) is FI w.r.t. system (1). Moreover, all the TDSs $S_{p.\tilde{h}(t)}$ generated by $\tilde{h}(t) = \rho h(t)$ with $\rho \in (0,1]$ are FI w.r.t. system (1), because $S_{p,\tilde{h}(t)} \subseteq S_{p,h(t)}$ and inequality (10a) written for $\tilde{H}(t) = \rho H(t)$ remains valid, proving that system (1) is locally $DI_{p,h(t)}$.

(b) Define the matrix

$$P(x,t) = H^{-1}(t)J(x,t)H(t) - H^{-1}(t)\dot{H}(t).$$

By using Lemma 1, we can show that

$$\mu_{|| \, ||_p}\left(M(x,t)\right) \le \int_0^1 \mu_{|| \, ||_p}\left(J(sx,t)\right) ds$$

since

$$\mu_{|| \, ||_p}\left(M(x,t)\right) = \lim_{\theta \downarrow 0}\left[\left\| \frac{1}{\theta} I + \left(H^{-1}(t)\int_0^1 J(sx,t) ds \, H(t) - \dot{H}(t)H^{-1}(t)\right)\right\|_p - \frac{1}{\theta}\right] =$$

$$= \lim_{\theta \downarrow 0} \left[\left\| \int_0^1 \left(\frac{1}{\theta} I + P(sx,t) \right) ds \right\|_p - \int_0^1 \frac{1}{\theta} ds \right] \leq \lim_{\theta \downarrow 0} \left[\int_0^1 \left\| \frac{1}{\theta} I + P(sx,t) \right\|_p ds - \int_0^1 \frac{1}{\theta} ds \right] =$$

$$= \lim_{\theta \downarrow 0} \int_0^1 \left[\left\| \frac{1}{\theta} I + P(sx,t) \right\|_p - \frac{1}{\theta} \right] ds = \int_0^1 \mu_{\|\,\|_p} \left(P(sx,t) \right) ds.$$

If inequality (10b) is true, then, $\forall t \in \mathbb{R}_+$, $\forall x \in S_{p,h(t)}$ we also have $\mu_{\|\,\|_p} \left(P(sx,t) \right) \leq 0$, $\forall s \in [0,1]$, yielding

$$\mu_{\|\,\|_p} \left(M(x,t) \right) \leq \int_0^1 \mu_{\|\,\|_p} \left(J(sx,t) \right) ds \leq 0.$$

The proof is completed since we use Part (a). □

Remark 2. If one of the inequalities (10a) or (10b) is met for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, then system (1) is globally $DI_{p,h(t)}$.

4. Diagonal Invariance of Linear Systems

Consider the time-invariant linear system

$$\dot{x}(t) = Cx(t). \tag{11}$$

The sufficient conditions for diagonal invariance given in the previous section become necessary and sufficient, as shown below, due to the linearity of system (11). The linear behavior also emphasizes the global meaning of the flow invariance property:

Lemma 3. *Given a positive vector function $h(t)$ (2) and an arbitrary Hölder p-norm, if TDS $S_{p,h(t)}$ (4) is FI w.r.t. system (11), then, for any constant $\varepsilon > 0$, the TDS $S_{p,\tilde{h}(t)}$ defined by $\tilde{h}(t) = \varepsilon h(t)$ is also FI w.r.t. system (11).*

Proof. For an arbitrary $\varepsilon > 0$, any trajectory $\tilde{x}(t)$ of system (11) initialized in $\tilde{x}(t_0) \in S_{p,\tilde{h}(t_0)}$ can be written as $\tilde{x}(t) = \varepsilon x(t)$, where $x(t)$ is the solution to (11) initialized in $x(t_0) = \tilde{x}(t_0)/\varepsilon \in S_{p,h(t_0)}$. The FI property of $S_{p,h(t)}$ implies $x(t) \in S_{p,h(t)}$, $\forall t > t_0$, which leads to $\tilde{x}(t) = \varepsilon x(t) \in S_{p,\varepsilon h(t)} = S_{p,\tilde{h}(t)}$, $\forall t > t_0$. □

Lemma 4. *Given a positive vector function $h(t)$ (2) and an arbitrary Hölder p-norm, TDS $S_{p,h(t)}$ (4) is FI w.r.t. system (11) if and only if function (6) is nonincreasing along each trajectory of system (11).*

Proof. *Sufficiency:* It is ensured by Lemma 1. *Necessity:* Given arbitrary $t_0 \in \mathbb{R}_+$ and $x_0 \in S_{p,h(t_0)}$, by using Lemma 3 for the trajectory $x(t) = x(t; t_0, x_0)$ of (11) with $\varepsilon = ||H^{-1}(t_0)x_0||_p$ we get $||(\varepsilon H(t))^{-1} x(t)||_p \leq 1$, or, equivalently,

$$||H^{-1}(t)x(t)||_p \leq \varepsilon = ||H^{-1}(t_0)x(t_0)||_p$$

for any $t > t_0$. The proof is completed, since t_0 and x_0 are arbitrary. □

Proposition 2. *Given a positive vector function* $h(t)$ (2) *and an arbitrary Hölder p-norm, the linear system* (11) *is* $DI_{p,h(t)}$ *if and only if* $\forall t \in \mathbb{R}_+$ *the following inequality is fulfilled:*

$$\mu_{||\ ||_p}\left(H^{-1}(t)C(H(t) - H^{-1}(t)\dot{H}(t)\right) \leq 0. \tag{12}$$

Proof. *Sufficiency:* It is ensured by Proposition 1 with $A(x,t) = J(x,t) = C$. *Necessity:* By using the notation $M(t) = H^{-1}(t)CH(t) - H^{-1}(t)\dot{H}(t)$, according to the Proof of Proposition 1, for arbitrary $t \in \mathbb{R}_+$, $\theta \in \mathbb{R}_+$, we can write

$$H^{-1}(t+\theta)\,x(t+\theta) = H^{-1}(t)\,x(t) + \theta M(t)\left(H^{-1}(t)x(t)\right) + \theta O(\theta)$$

with $\lim\limits_{\theta\downarrow 0} ||O(\theta)||_p = 0$. On the other hand, the following inequalities hold true:

$$\mu_{||\ \ ||_p}(M(t)) = \lim_{\theta\downarrow 0} \frac{\sup\limits_{||H^{-1}(t)x(t)||_p=1} ||(I+\theta M(t))H^{-1}(t)x(t)||_p - 1}{\theta} =$$

$$= \lim_{\theta\downarrow 0}\left[\frac{1}{\theta}\sup_{||H^{-1}(t)x(t)||_p=1}\left(||H^{-1}(t)x(t) + \theta M(t)H^{-1}(t)x(t)||_p\right) - \frac{1}{\theta}\right] =$$

$$= \lim_{\theta\downarrow 0}\left[\frac{1}{\theta}\sup_{||H^{-1}(t)x(t)||_p=1}\left(||H^{-1}(t+\theta)x(t+\theta) - \theta O(\theta)||_p\right) - \frac{1}{\theta}\right] \leq$$

$$= \lim_{\theta\downarrow 0}\left[\frac{1}{\theta}\sup_{||H^{-1}(t)x(t)||_p=1}\left(||H^{-1}(t+\theta)x(t\ +\theta)||_p + \theta||O(\theta)||_p\right) - \frac{1}{\theta}\right] \leq$$

$$\leq \lim_{\theta\downarrow 0}\left(\frac{1}{\theta} + ||O(\theta)||_p - \frac{1}{\theta}\right) = 0$$

since

$$||H^{-1}(t+\theta)x(t+\theta)||_p \leq\ ||H^{-1}(t)x(t)||_p = 1,$$

according to Lemma 4. □

Remark 3. The concrete expression of the matrix measure used in inequality (12) allows deriving particular forms for the $DI_{p,h(t)}$ condition relative to different Hölder p-norms, as follows.

For the Hölder norm $p = \infty$, inequality (12) is equivalent to the following n differential inequalities

$$DI_{\infty,h(t)}: \qquad c_{ii}h_i(t) + \sum_{j=1, j\neq i}^{n} |c_{ii}|h_j(t) \leq \dot{h}_i(t),\ \ i = 1,\ldots,n. \tag{13}$$

Actually, the $DI_{\infty,h(t)}$ condition (13) was obtained in [15] directly from the subtangency condition (see, e.g [14]) applied to time-dependent, symmetrical rectangular sets.

For the Hölder norm $p = 1$, the approach is *mutatis mutandis* similar to the one for the Hölder norm $p = \infty$.

For the Hölder norm $p = 2$, inequality (12) is equivalent to the matrix differential inequality

$$DI_{2,h(t)}: \qquad C^T H^{-2}(t) + H^{-2}(t)C + \frac{d}{dt}\left(H^{-2}(t)\right) \le 0, \tag{14}$$

written in the sense of semidefinite negativeness. In connection with Remark 1, note that a solution $H^{-1}(t)$ to (14) does not necessarily define a weak Lyapunov function for the linear system (11) when it is replaced in $W(x,t)$ defined by (6). □

5. A Comparison Theorem for Diagonal Invariance

Proposition 1 is not easy to handle as a practical instrument for systems of higher dimension and, therefore, some results with a larger applicability, handling simplified sufficient conditions, are presented in the sequel. The purpose is to avoid the usage of the matrices $A(x,t)$, $J(x,t)$ appearing in inequalities (10a), (10b), by operating with constant matrices that majorize $A(x,t)$, $J(x,t)$.

Given a square matrix $Q = (q_{ij})$, $i,j = 1,\dots,n$, denote by $\overline{Q} = (\bar{q}_{ij})$, $i,j = 1,\dots,n$, a majorant of Q, built as an *essentially nonnegative matrix*, i.e. a matrix with nonnegative off-diagonal elements [1]:

$$\begin{aligned} &\bar{q}_{ii} = q_{ii}, \ i = 1,\dots,n; \\ &\bar{q}_{ij} = |q_{ij}|, \ i \neq j, \ i,j = 1,\dots,n. \end{aligned} \tag{15}$$

Theorem 1. *Given a positive vector function $h(t)$ (2) and an arbitrary Hölder p-norm, consider the set $\Omega_p \supseteq \bigcup_{t\in\mathbb{R}_+} S_{p,h(t)}$. Assume that one of the following two conditions is satisfied,*

$$(a) \qquad \forall t \in \mathbb{R}_+, \forall x \in \Omega_p : \overline{A(x,t)} \le C, \tag{16a}$$

$$(b) \qquad \forall t \in \mathbb{R}_+, \forall x \in \Omega_p : \overline{J(x,t)} \le C. \tag{16b}$$

If the linear system (11) is $DI_{p,h(t)}$, then the time-variant nonlinear system (1) is $DI_{p,h(t)}$.

In order to prove Theorem 1, let us give a technical result referring to essentially nonnegative matrices.

Lemma 5. *Consider a square matrix $Q = (q_{ij})$, $i,j = 1,\dots,n$, its majorant $\overline{Q} = (\bar{q}_{ij})$, $i,j = 1,\dots,n$, built according to (15), and an essentially nonnegative matrix P that satisfies the componentwise matrix inequality $\bar{Q} \le P$. For any Hölder p-norm, the following inequalities hold:*

$$\mu_{||\ ||_p}(Q) \le \mu_{||\ ||_p}(\bar{Q}) \le \mu_{||\ ||_p}(P). \tag{17}$$

Proof. It results from the definition of the matrix measures and the monotonicity of the matrix norms induced by Hölder vector norms.

(i) First let us show that if M and N are two nonnegative $n \times n$ matrices satisfying $M \le N$, then $||M||_p \le ||N||_p$. Indeed, due to the continuity of the function $||My||_p$ for y in the closed hyper-ball $\mathcal{B}_{p,1} = \{y \in \mathbb{R}^n |\ ||y||_p = 1\}$ there exists $y^* \in \mathcal{B}_{p,1}$, $y^* \ge 0$, so that

$$||M||_p = \sup_{y\in\mathcal{B}_{p,1}} ||My||_p = ||My^*||_p \le ||Ny^*||_p \le ||N||_p||y^*||_p = ||N||_p.$$

(ii) Next, in order to prove the second inequality in (17), for the given matrix Q let us consider $\theta > 0$ sufficiently small, so that matrix $\frac{1}{\theta}I + \bar{Q}$ is nonnegative (e.g. $\theta < \theta^*$ with $1/\theta^* > q_{ii}$, $i = 1, \ldots, n$). If matrix P satisfies $\bar{Q} \le P$, then, according to part (i) of the proof, $\left\|\frac{1}{\theta}I + \bar{Q}\right\|_p \le \left\|\frac{1}{\theta}I + P\right\|_p$, leading to

$$\mu_{\| \,\|_p}(\bar{Q}) = \lim_{\theta \downarrow 0}\left[\left\|\frac{1}{\theta}I + \bar{Q}\right\|_p - \frac{1}{\theta}\right] \le \lim_{\theta \downarrow 0}\left[\left\|\frac{1}{\theta}I + P\right\|_p - \frac{1}{\theta}\right] = \mu_{\| \,\|_p}(P)\,.$$

(iii) The first inequality in (17) results in a similar manner, taking into account that for $\theta > 0$ sufficiently small, matrix $\frac{1}{\theta}I + \bar{Q}$ is nonnegative and there exists $y^* \in \mathcal{B}_{p,1}$ so that the following relation holds:

$$\left\|\frac{1}{\theta}I + Q\right\|_p = \left\|\left(\frac{1}{\theta}I + Q\right)y^*\right\|_p = \left\|\left(\frac{1}{\theta}I + \bar{Q}\right)|y^*|\right\|_p \le \left\|\left(\frac{1}{\theta}I + \bar{Q}\right)\right\|_p,$$

leading to $\mu_{\| \,\|_p}(Q) \le \mu_{\| \,\|_p}(\bar{Q})$. □

Proof of Theorem 1. If system (11) is $DI_{p,h(t)}$, then Proposition 2 ensures the fulfillment of (12).

(a) If condition $(16a)$ is valid, then the following componentwise matrix inequality holds true

$$\overline{H^{-1}(t)\,A(x,t)\,H(t) - H^{-1}(t)\,\dot{H}(t)} \;\le\; H^{-1}(t)\,C\,H(t) - H^{-1}(t)\,\dot{H}(t)$$

and Lemma 5 ensures that (12) implies $(10a)$. Then apply Proposition $1(a)$ to show that system (1) is $DI_{p,h(t)}$.

(b) Similarly, if condition $(16b)$ is valid, then

$$\overline{H^{-1}(t)\,J(x,t)\,H(t) - H^{-1}(t)\,\dot{H}(t)} \;\le\; H^{-1}(t)\,C\,H(t) - H^{-1}(t)\dot{H}(t)$$

and Lemma 5 ensures that (12) implies $(10a)$. Then apply Proposition $1(b)$ to show that system (1) is $DI_{p,h(t)}$. □

Remark 4. The proof of Theorem 1 shows that the practical usage of the comparison method requires testing the fulfillment of inequality (12) for the linear system (11).

6. An Illustrative Example

Consider the nonlinear system

$$\dot{x}(t) = Bx(t) + Wg(x(t)), \tag{18}$$

where $B = \mathrm{diag}\{b_1, \ldots, b_n\}$, $b_i < 0$, $i = 1, \ldots, n$, $W \in \mathbb{R}^{n\times n}$.

The function $g(x) = [g_1(x) \cdots g_n(x)]^T : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on $\mathbb{R}^n$ and fulfills $g_i(x) = g_i(x_i)$, $g_i(0) = 0$ and $0 \le g_i'(r) \le L_i$, $\forall r \in \mathbb{R}$, $i = 1, \ldots, n$.

This type of systems characterizes the dynamics of recurrent neural networks without delay, described with respect to the EP $\{0\}$.

Define the matrix

$$\Pi = B + \widetilde{W}\Lambda, \tag{19}$$

where the elements of $\widetilde{W} = [\tilde{w}_{ij}] \in \mathbf{R}^{n\times n}$ are given by $\tilde{w}_{ii} = \max\{0, w_{ii}\}$, $i = 1, \ldots, n$, $\tilde{w}_{ij} = |w_{ij}|$, $i \neq j$, $i, j = 1, \ldots, n$, and $\Lambda = \mathrm{diag}\{L_i, \cdots, L_n\}$.

Corollary 1. *Given a positive vector function $h(t)$ (2) and an arbitrary Hölder p-norm, if the linear system*

$$\dot{x}(t) = \Pi\, x(t), \tag{20}$$

defined with matrix Π (19)*, is globally $DI_{p,h(t)}$, then the nonlinear system* (18) *is globally $DI_{p,h(t)}$.*

Proof. Take into account that, the Jacobian matrix of the function $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = Bx + Wg(x)$, fulfills

$$\overline{J(x)} = \overline{B + W\mathrm{diag}\{g_1'(x_1), \cdots, g_n'(x_n)\}} \leq B + \widetilde{W}\Lambda = \Pi$$

for all $x \in \mathbb{R}^n$ and apply Theorem 1(b) with $C = \Pi$. □

The practical usage of Corollary 1 means testing the fulfillment of inequality (12) with $C = \Pi$, in accordance with Remark 4. In the particular case of the Hölder norm with $p = \infty$, (12) is equivalent with the differential inequality

$$\dot{h}(t) \geq \Pi\, h(t), \tag{21}$$

condition previously obtained in [10] by applying the subtangency theory presented in [14].

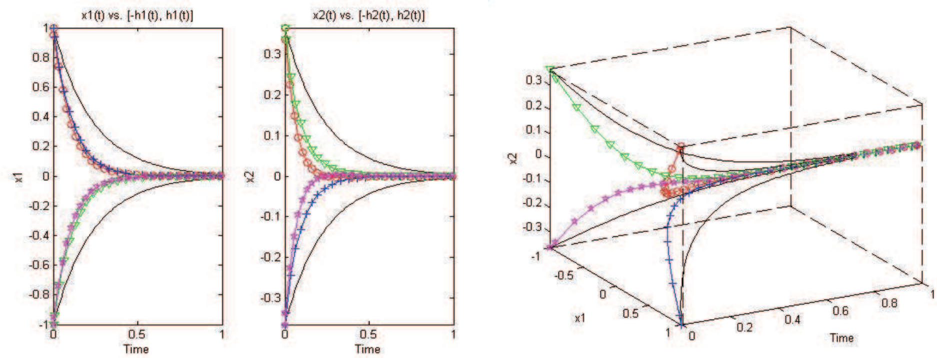
The following numerical example, adapted from [10], refers to the nonlinear system described by (18) with $B = \mathrm{diag}\,\{5,\ 7\}$, $W = \begin{bmatrix} -5 & -1 \\ -1 & -3.5 \end{bmatrix}$, $g_1(x_1) = \mathrm{tansig}(x_1)$ and $g_2(x_2) = \mathrm{tansig}(2x_2)$. The numerical value of matrix Π constructed according to (19) with $\Lambda = \mathrm{diag}\,\{1,\ 2\}$ is $\Pi = \begin{bmatrix} -5 & 2 \\ 1 & -7 \end{bmatrix}$.

Consider the vector function $h : \mathbb{R}_+ \to \mathbb{R}^2$, $h(t) = e^{\sigma t} d$, defined with σ the *importance* eigenvalue of Π [6], $\sigma = \lambda_{\max}(\Pi) = -6 + \sqrt{3} < 0$, and d its corresponding eigenvector satisfying $||d||_\infty = 1$, $d = \begin{bmatrix} 1 & \left(\sqrt{3} - 1\right)/2 \end{bmatrix}^T$ (which is positive). Since $h(t)$ satisfies relation (21) as an equality, the considered nonlinear system is globally $DI_{\infty,h(t)}$.

Figure 1 depicts the symmetrical TDS $S_{\infty,h(t)}$ (solid lines), as well as the four state-trajectories (marked lines) initialized at $t_0 = 0$ in the vertices of $S_{\infty,h(0)}$. Obviously, all the state-trajectories initialized in $S_{\infty,h(0)}$ remain inside $S_{\infty,h(t)}$, $t \geq 0$.

7. Conclusions

The framework we have developed for the analysis of flow invariance covers a large class of dynamical systems and allows considering invariant sets depending on time. The results presented by this paper operate as sufficient conditions in the general case of time-variant nonlinear systems, but, for linear systems, they provide necessary and sufficient conditions. The



(a) (b) (c)

Fig. 1. Plots corresponding to four state-trajectories of the nonlinear system:
(a) 2D visualization for state variable x_1 versus time;
(b) 2D visualization for state variable x_2 versus time;
(c) 3D visualization.

approach to nonlinear systems is built as a comparison theorem whose practical tractability is ensured by the convenient handling of the $DI_{p,h(t)}$ characterization for the linear comparison system. Our framework creates a unified point of view on $DI_{p,h(t)}$ testing, which is able to accommodate results of previous works as particular cases. The adaptation of this framework to discrete-time dynamics is a straightforward task.

References

[1] A. Bermann and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994.

[2] F. Blanchini, and S. Miami. *Set-Theoretic Methods in Control*, Birkhäuser, Boston, 2008.

[3] S. Boyd, E. Feron, L. El Ghaoui and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[4] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[5] Y. Fang and T.G. Kincaid, *Stability analysis of dynamical neural networks*, IEEE Trans. Neural Networks, vol. 7 (1996), pp. 996-1006.

[6] L.T. Gruyitch, J.P. Richard, P. Borne and J.C. Gentina, *Stability Domains (Nonlinear Systems in Aviation, Aerospace, Aeronautics and Astronautics)*, Chapman & Hall/CRC, London, 2004.

[7] A. Hmamed, *Componentwise stability of 1-D and 2-D linear discrete systems*, Automatica, vol. 33 (1997), pp. 1759-1762.

[8] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.

[9] E. Kaszkurewicz and A. Bhaya, *Matrix Diagonal Stability in Systems and Computation*, Birkhauser, Boston, 2000.

[10] M.H. Matcovschi and O. Pastravanu, *Flow-invariance and stability analysis for a class of nonlinear systems with slope conditions*, Eur. J. Control, vol. 10 (2004), pp. 352-364.

[11] A. Michel and K. Wang, *Qualitative Theory of Dynamical Systems*, Marcel Dekker, New York, 1995.

[12] O. Pastravanu and M. Voicu, *Interval Matrix Systems - Flow Invariance and Componentwise Asymptotic Stability*, Differential and Integral Equations, vol. 15 (2002), pp. 1377–1394.

[13] O. Pastravanu and M. Voicu. *Dynamics of a class of uncertain nonlinear systems under flow-invariance constraints*, Int. Journ. of Math. and Math. Scie., USA, vol. 2003 (2003), pp. 263-294.

[14] H.N. Pavel, *Differential Equations: Flow Invariance and Applications*, Research Notes in Mathematics, No. 113, Pitman, Boston, 1984.

[15] M. Voicu, *Free response characterization via flow-invariance*, Prep. 9-th World Congress of IFAC, Budapest, vol. 5, pp. 12-17 (1984).

[16] M. Voicu, *On the application of the flow-invariance method in control theory and design*, Prep. 10-th World Congress of IFAC, München, vol. 8, pp. 364–369 (1987).