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Topological Degree for Pseudomonotone Potential Operators

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Abstract. In this note we give an extension of a recent result concerning the fact that the topological degree of a coercive map is one still remains true for a pseudomonotone potential map between a reflexive Banach space and its dual space. Some considerations concerning the solution property of degree required in applications are also appended.

Keywords: topological degree, potential operators, pseudomonotone maps.

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1 Introduction

Let U be an open set of some real Hilbert space H and suppose that the gradient $\nabla u : U \to H$ of a given function $f \in C^1(U, \mathbb{R})$ is a Leray-Schauder map (LS-map for short), that is, $\nabla f = I - F$, where $I : H \to H$ is the identity operator and $F \in K(U, H)$ is a compact map. Let $B(x_0, r)$ be the open ball of center x_0 and radius r, and a bar denotes the closure in H.

In [1] Amann proved the following results:

Theorem 1 *Suppose that, for some $\beta \in \mathbb{R}$, the set $V := f^{-1}(-\infty, \beta)$ is bounded and $\overline{V} \subset U$. Moreover, suppose that there exist numbers $\alpha < \beta$, $r > 0$ and a point $x_0 \in U$ such that $f^{-1}(-\infty, \alpha] \subset \overline{B}(x_0, r) \subset V$ and $\nabla f(x) \neq 0, \forall x \in f^{-1}[\alpha, \beta]$. Then $d_{LS}(\nabla f, B(0, r), 0) = 1$.*

Direct consequences:

(C1) *Suppose that $U = H$ and $f(x) \to \infty$ as $\|x\| \to \infty$. If $\nabla f(x) \neq 0$ for $\|x\| \geq r_0$, with some $r_0 > 0$, then there exists $r_1 \geq r_0$ such that $d_{LS}(\nabla f, B(0, r), 0) = 1, \forall r \geq r_1$.*

(C2) *If $x_0 \in U$ is an isolated critical point of f at which f has a local minimum, then LS-index of ∇f at x_0, $i_{LS}(\nabla f, x_0) = \lim_{s \to 0} d_{LS}(\nabla f, B(x_0, s), 0) = 1$.*

(C3) *Under the hypotheses of theorem 1, suppose that $x_1 \in V$ is a critical point of f, which is not a global minimum of f in V, such that either F is Fréchet differentiable at x_1 and $\lambda = 1$ is not an eigenvalue of $F'(x_1) \in \mathcal{L}(H)$, or x_1 is a local minimum. Then f has at least three critical points in V. The assertion remains true in case $U = H$ and $f(x) \to \infty$ as $\|x\| \to \infty$.*

In [3], D.M. Duc et al. extend these results for continuous potential operators of class (S_+) by using Galerkin method as Browder first in [2].

In this note we extend the above results to a class of pseudomonotone potential operators by using an approach due to Berkowritz and Mustonen (see for details [7]).

2 Topological Degree for Pseudomonotone Operators

Let X be a real reflexive Banach space and X^* its dual space, whose norms are denoted by the same symbol $\|\cdot\|$, without danger of confusion. By "$\to$" and "$\rightharpoonup$", we denote the *strong* and *weak convergence* respectively, while $\langle\cdot,\cdot\rangle$ means the *duality pairing between* X *and* X^*.

The key role in definition of the topological degree for mappings of monotone type is played by the so called *mappings of type* (S_+) and the *pseudomonotone* ones. The operator T is of type (S_+), denoted $T \in (S_+)$, if each sequence $\{u_n\} \subset X$ with $u_n \rightharpoonup u_0$ for which

$$\limsup < Tu_n, u_n - u_0 > \leq 0 \tag{1}$$

is in fact strongly convergent in X.

The operator T is *pseudomonotone*, denoted $T \in (PM)$, if each sequence $\{u_n\}$ in X with $u_n \rightharpoonup u_0$ for which (1) holds, it follows that $< Tu_n, u_n - u_0 > \to 0$ and $Tu_n \rightharpoonup Tu_0$. Moreover, T is *quasimonotone*, denoted $T \in (QM)$, if for each sequence $\{u_n\}$ in X with $u_n \rightharpoonup u_0$, it follows that

$$\limsup < Tu_n - Tu_0, u_n - u_0 > \geq 0.$$

We assume that all mappings considered are *bounded*, i.e., they carry bounded sets into bounded sets, and *demicontinuous*, i.e., $Tu_n \rightharpoonup Tu_0$ for any $u_n \to u_0$.

The original construction of a (S_+)-degree theory, performed independently by I.V. Skrypnik and F.E. Browder, were based on Galerkin approximations and Brouwer degree. Our approach for the degree of monotone operators relies on the LS-degree and the Browder-Ton representation theorem (see [4], p. 302):

Theorem 2 *Let X be a separable Banach space and let S be a countable set in X. Then there exists a separable Hilbert space H and a compact one-to-one linear operator $\psi : H \to X$ such that $S \subset \psi(H)$ and $\psi(H)$ is dense in X.*

Define now the adjoint operator $\varphi : X^* \to H$ by

$$(\varphi(w), v) =< w, \psi(v) >, \forall v \in H, w \in X^*. \tag{2}$$

Clearly, φ is also a linear compact injection. Here $(\cdot,\cdot)$ is the scalar product in H which induces the norm $|\cdot|$ and we can suppose $\|\psi\| = |\varphi| = 1$ without loss of generality.

For a given open bounded set $D \subset X$, let us denote

$$F_D(S_+) := \{F : \overline{D} \to X^* | F \in (S_+), \textit{bounded and demicontinuos}\}.$$

For each $F \in F_D(S_+)$ we can associate the family of mappings $\{F_\varepsilon | \varepsilon > 0\}$ defined by

$$F_\varepsilon(u) := u + \frac{1}{\varepsilon}\psi\varphi F(u), u \in \overline{D}. \tag{3}$$

Observe that, for any fixed $\varepsilon > 0$, F_ε maps $\overline{D}$ into X and it has the form $I - T_\varepsilon$, where $T_\varepsilon = -\frac{1}{\varepsilon}\psi\varphi F$ is compact. Hence $F_\varepsilon \in LS(D, X)$ and $d_{LS}(F_\varepsilon, D, p)$ is well defined for all $p \notin F_\varepsilon(\partial D)$. Therefore, for any $p \in X^*$ with $p \notin F(\partial D)$ we can define (S_+)-degree as follows:

$$d_{S_+}(F, D, p) = \lim_{\varepsilon \to 0} d_{LS}(F_\varepsilon, D, p). \tag{4}$$

Of course, the (S_+)-degree has all the properties of LS-degree, except the normalization property, when we take instead of I the duality map $J : X \to 2^{X^*}$, defined by $\|J(x)\| = \|x\|$ and $< J(x), x >= \|x\|^2$. For X a locally uniformly convex Banach space, J is an operator (an univalent mapping), $J : X \to X^*$, and belongs to class (S_+) as we can easily see ([4], p.145).

The preceding construction facilitate to introduce a degree for the more general class $F_D(QM)$ of quasimonotone operators because of the following result due to Calvert and Well (see [6], p.17)

Lemma 1 *A demicontinuos operator $T : X \to X^*$ is quasimonotone if and only if for each $\mu > 0$ the mapping $T + \mu J$ is of type (S_+).*

Therefore the *(QM)*-degree can be obtained through the approximation

$$d_{QM}(F, D, P) = \lim_{\mu \to 0} d_{S_+}(F + \mu J, D, p) \tag{5}$$

with $p \notin \overline{F(\partial D)}$. But this degree can not be a classical one, because in this case the image $F(A)$ of a closed set $A \subseteq \overline{D}$ is no more closed. This fact modifies all the properties; for instance, *the solution property* (see [7] for details) will be replaced in this case by the following implication

$$d_{QM}(F, D, p) \neq 0 \Rightarrow p \in \overline{F(D)}. \tag{6}$$

Since $(S_+) \subset (PM) \subset (QM)$, the *(QM)*-degree is well defined for all $F \in F_D(PM)$. The interest of pseudomonotone mappings in applications is due to the following closeness property

(PM) *For each $F \in F_D(PM)$, the set $F(A)$ is closed whenever $A \subseteq \overline{D}$ is weakly closed.*

Indeed, if $\{w_n\} \subset F(A)$ with $w_n \to w$, then $w_n = F(u_n)$ for some $\{u_n\} \subset A$. Since D is bounded, $u_n \rightharpoonup u$ for some $u \in A$ at least on a subsequence. Thus, $\limsup < F(u_n), u_n - u >= 0$, implying $F(u_n) \rightharpoonup F(u)$ and hence $w = F(u) \in F(A)$.

In particular, if D in convex, then $\overline{D}$ is weakly closed, implying that $F(\overline{D})$ is closed. Consequently, for $F \in F_D(PM)$ and D convex, we can conclude

$$d_{QM}(F, D, p) \neq 0 \Rightarrow p \in F(\overline{D}). \tag{7}$$

This justifies the application of the solution property for pseudomonotone mappings on convex sets, and assures to our extension from the next section, the maximum topological meaning.

3 Topological Degree for Pseudomonotone Potential Operators

Let U be an open set in a real reflexive Banach space X and $f \in C^1(U, R)$ a given function with $F = \nabla f : X \to X^*$, the Gâteaux gradient, a pseudomonotone operator. Observe that we can work with Gâteaux gradients instead of Fréchet ones (see e.g. [8]).

Theorem 3 *Suppose that for some $\beta \in \mathbb{R}$, the set $V = f^{-1}(-\infty, \beta)$ is bounded and $\overline{V} \subset U$. Moreover suppose that there are numbers $\alpha < \beta$ and $r > 0$ and a point $x_0 \in U$ such that $f^{-1}(-\infty, \alpha] \subset B(x_0, r) \subset V$ and $F(x) \neq 0, \forall x \in f^{-1}[\alpha, \beta]$. Then $d_{PM}(F, V, 0) = 1$.*

Proof. Since $V = f^{-1}(-\infty, \beta)$ is bounded and $\overline{V} \subset U$ it follows that $\mathcal{V} = \psi^{-1}(V)$ is relatively compact in H and thus there exist $u_0 \in H$ and $r > 0$ such that $\mathcal{V} \subset B(u_0, r)$. Similarly, $\psi(B(u_0, r))$ is relatively compact in X and $\overline{V} \subset U$, which implies that $\psi(B(u_0, r)) \subset B(x_0, r) \subset U$ for $x_0 = \psi(u_0)$ and some $R > 0$.

Now, for any $\alpha < \beta$ the set $V_\alpha := f^{-1}(-\infty, \alpha] \subset V$ and thus it is bounded and closed. Then $\psi^{-1}(V_\alpha)$ is compact and $\psi^{-1}(V_\alpha) \subset B(u_0, r)$. Hence $V_\alpha \subset V \subset B(x_0, R) \subset U$.

Suppose that $F(x) \neq 0, \forall x \in f^{-1}[\alpha, \beta]$. Then $\mathcal{F} := \varphi F \psi : H \to H$ is compact in H and

$$|\mathcal{F}(u)| = \sup_{|v|=1} (\mathcal{F}(u), v) = \sup_{|v|=1} (\varphi F \psi(u), v) = \sup_{|v|=1} < F\psi(u), \psi(v) >=$$
$$= \|F(\psi u)\| \cdot \|\psi\| = \|F(x)\| > 0, \qquad \forall x = \psi u \in f^{-1}[\alpha, \beta]. \tag{8}$$

As $F \in (PM) \Rightarrow (F + \mu J) \in (S_+)$ and thus $\mathcal{F}_{\varepsilon\mu} := I + \frac{1}{\varepsilon}\varphi(F + \mu J)\psi \in LS(H)$ and $d_{LS}(\mathcal{F}_{\varepsilon\mu}, \mathcal{V}, 0) = 1$ by theorem 1. Hence $d_{S_+}(F + \mu J, V, 0) = \lim_{\varepsilon \to 0} d_{LS}(\mathcal{F}_{\varepsilon\mu}, \mathcal{V}, 0) = 1$ and $d_{PM}(F, V, 0) = \lim_{\mu \to 0}(F + \mu J, V, 0) = 1$ by definition.□

Remark that the result is generally true for mappings of class *(QM)*, but we are restricted to the class *(PM)* because of the solution property as stated in the previous section.

We shall prove that all direct consequences stated in Section 1 still remain true also for operators of class *(PM)*. Here we present the Amann's proofs adapted to our case.

Corollary 1 *Let f be a real C^1-function on X such that its Gâteaux gradient $F = \nabla f : X \to X^*$ is a bounded map of class (PM). Suppose that $f(x) \to \infty$ as $\|x\| \to \infty$ and that $F(x) \neq 0$ for all $\|x\| > r_0$ and some $r_0 > 0$. Then there exists $r_1 \geq r_0$ such that $d_{PM}(F, B(0, r), 0) = 1, \forall r \geq r_1$.*

Proof. Observe that $f(x) = \int_0^1 < F(sx), x > ds$. Take $x = \psi u$ and write

$$f(\psi u) = (f\psi)(u) = \int_0^1 < F(s\psi u), \psi u > ds =$$
$$= \int_0^1 (\varphi F \psi(su), u) ds = \int_0^1 (\mathcal{F}(su), u) ds \tag{9}$$

with $\mathcal{F}$ a compact operator on H. Here $f\psi$ maps bounded sets from H into bounded sets in $\mathbb{R}$ and $\psi(B(0, r))$ is a compact set in X because $\psi : H \to X$ is a compact linear map. Hence there exists an $r_0 > 0$ such that $\psi(B(0, \delta)) \subseteq B(0, r_0) \subset X$.

Thus, let $\alpha = \sup f(B(0, r_0))$ and $r_1 = sup\{\|x\| \,; x \in f^{-1}((-\infty, \alpha])\}$. For given $r \geq r_1$, fix $\beta > \sup f(B(0, r))$ and apply theorem 3 with $x_0 = 0$ and the excision property of degree. □

Corollary 2 *Suppose that $f : X \to \mathbb{R}$ is a C^1 function on an open convex set $U \subseteq X$ with its Gâteaux gradient $F = \nabla f : X \to X^*$ of class (PM) on U. Let $x_0 \in U$ be an isolated critical point of f on U. If f has a local minimum at x_0, then $i(F, x_0) = 1$.*

Proof. Since $F \in (PM)$ it follows that f is weakly sequentially lower semicontinuous on U (see [4], p.24). We can suppose without loss of generality that $x_0 = 0$, $f(0) = 0$ and there exists an $r_0 > 0$ such that $U = B(0, r_0)$ and 0 is the unique critical point of f in U.

We claim that

$$\inf f(\overline{B}(0, r_2) \setminus B(0, r_1)) > 0 \tag{10}$$

whenever $0 < r_1 \leq r_2 < r_0$.

Indeed, otherwise it would exist a sequence $\{x_n\} \subset \overline{B}(0, r_2) \setminus \overline{B}(0, r_1)$ such that $f(x_n) \to 0$. As X is a reflexive Banach space and $\{x_n\}$ is bounded we can assume that $x_n \rightharpoonup x \in \overline{B}(0, r_2)$ at least on a subsequence. Since f is weakly sequentially lower semicontinuous it results the implication

$$0 \leq f(x) \leq \liminf f(x_n) = 0 \Longrightarrow x = 0. \tag{11}$$

On the other hand, for any $n \in \mathbb{N}$ by the Lagrange's formula there is an $s_n \in (0, 1)$ such that

$$f(x_n) - f\left(\frac{x_n}{2}\right) =< F\left(\frac{x_n}{2} + s_n \frac{x_n}{2}\right), \frac{x_n}{2} > .$$

Since $f\left(\frac{x_n}{2}\right) \geq 0$ and $\lim f(x_n) = 0$, by (11) we get

$$0 \geq \limsup < F\left(\frac{x_n}{2} + s_n \frac{x_n}{2}\right), \frac{x_n}{2} >= \limsup < F\left((1 + s_n)\frac{x_n}{2}\right), (1 + s_n)\frac{x_n}{2} - s_n \frac{x_n}{2} > .$$

As $F \in (PM)$, it follows that $F + \mu J \in (S_+)$ and

$$\begin{aligned}
\limsup < F\left((1 + s_n)\frac{x_n}{2}\right) + \mu J\left((1 + s_n)\frac{x_n}{2}\right), (1 + s_n)\frac{x_n}{2} - s_n\frac{x_n}{2} >= \\
= \limsup < F\left((1 + s_n)\frac{x_n}{2}\right), (1 + s_n)\frac{x_n}{2} - s_n\frac{x_n}{2} > + \\
+\mu \limsup < J\left((1 + s_n)\frac{x_n}{2}\right), (1 + s_n)\frac{x_n}{2} - s_n\frac{x_n}{2} >\leq \\
\leq \mu \limsup \left[< J\left((1 + s_n)\frac{x_n}{2}\right), (1 + s_n)\frac{x_n}{2} > - < J\left((1 + s_n)\frac{x_n}{2}\right), s_n\frac{x_n}{2} >\right] \leq \\
\leq \mu \limsup \left[\left\|(1 + s_n)\frac{x_n}{2}\right\|^2 - \left\|J\left((1 + s_n)\frac{x_n}{2}\right)\right\| \cdot \left\|s_n\frac{x_n}{2}\right\|\right] \leq \\
\leq -\mu \limsup \left\|(1 + s_n)\frac{x_n}{2}\right\| \cdot \left\|s_n\frac{x_n}{2}\right\| \leq 0.
\end{aligned}$$

Hence $(1 + s_n)\frac{x_n}{2} \to 0$ and also $x_n \to 0$, in contradiction with $\|x_n\| \geq r_1$.

Now, fix r_1 and r_2 with $0 < r_1 < r_2 < r_0$, take $\beta = \inf f(\overline{B}(0, r_2) \setminus B(0, r_1))$ and choose $r > 0$ such that $\overline{V} = \overline{B}(0, r) \subset f^{-1}(-\infty, \beta)$. The result follows by applying theorem 3 with $U = B(0, r_2)$ and $\alpha = \frac{1}{2}\inf f(\overline{B}(0, r_2) \setminus B(0, r))$ since $i_{PM}(F, 0) = d_{PM}(F, V, 0)$. □

Corollary 3 *Under the hypotheses of theorem 3, suppose that $x_1 \in V$ is a critical point of f, which is not a global minimum of f in V, such that either F is Fréchet differentiable at x_1 and $\lambda = 1$ is not an eigenvalue of $F'(x_1) \in \mathcal{L}(X, X^*)$ or x_1 is a local minimum. Then f has at least three critical points in V. The assertion remains true in the case $U = X$ and $f(x) \to \infty$ as $\|x\| \to \infty$.*

Proof. As $F \in (PM) \implies (F + \mu J) \in (S_+)$ and thus $\mathcal{F}_\mu = \varphi(F + \mu J)\psi$ is compact and also of (S_+) type on H. By chain rule and the Krasnoselsskii's theorem $\mathcal{F}'_\mu(u_1) = \varphi'(F + \mu J)'\psi'(u_1) = \varphi(F + \mu J)'(x_1)$ (with $x_1 = \psi(u_1)$) is a compact linear operator on H and, thus, it is a continuous automorphism of H because $\lambda = 1$ is not an eigenvalue of $\mathcal{F}'_\mu(u_1)$. Hence x_1 is an isolated zero of F and $i(\varphi F, x_1) = \pm 1$ by the Leray-Schauder index formula.

Since $F \in (PM)$ it follows that f is weakly sequentially lower semicontinous on V and it attains its global minimum at some $x_2 \in V$ by the Weienstrass theorem (see [4], p. 24) and $x_1 \neq x_2$. If x_1 and x_2 are the only critical points in V then

$$1 = d_{LS}(\varphi F, V, 0) = i(\varphi F, x_1) + i(\varphi F, x_2) = 0 \text{ or } 2$$

by the additivity theorem of LS-degree (see [5], p. 73), which is a contradiction. Thus, f must have at least three critical points in V. The last statement follows by Corollary 1. □

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