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Soldered Tensor Fields of Normalized Submanifolds

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Dedicated to the centenary of the Mathematical Seminar "Al. Myller", Jassy, Romania and to Acad. Prof. Constantin Corduneanu on his 80-eth anniversary

Abstract. Soldered forms, multivector fields and Riemannian metrics were studied in our earlier paper [2]. In particular, it was shown that a Riemannian submanifold is totally geodesic iff the metric is soldered to the submanifold. In the present paper, we discuss general, soldered tensor fields. In particular, we prove that the almost complex structure of an almost Kähler manifold is soldered to a submanifold iff the latter is an invariant, totally geodesic submanifold.

Keywords: normalized submanifold, soldered tensor field.

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1 Introduction

In the present paper, all the manifolds, mappings, bundles, tensor fields, etc. are differentiable of class C^∞ and we use the standard notation of Differential Geometry, including the Einstein summation convention. The reader may consult [1] for all the differential geometric notion and results that are used in the paper.

If N^n is a submanifold of M^m (indices denote dimension), then a *normalization* of N by a *normal bundle* νN is a splitting

$$TM|_N = TN \oplus \nu N \tag{1.1}$$

(T denotes tangent bundles). A submanifold endowed with a normalization is called a *normalized submanifold* and a vector field X on M is tangent or normal to N if $X|_N$ belongs to $TN, \nu N$, respectively. The best known case is that of a Riemannian normalization $\nu N = T^{\perp_g} N$, where g is a Riemannian metric on M. In fact, given an arbitrary normalization, it is easy to construct metrics g such that the normalization is g-Riemannian. Similarly, if N is a symplectic submanifold of a symplectic manifold (M, ω), $\nu N = T^{\perp_\omega} N$ defines the *symplectic normalization*. Another interesting example is that of a submanifold N such that, $\forall x \in N$, $T_xM = T_xN \oplus T_x\mathcal{F}$, where $\mathcal{F}$ is a foliation of M; then we may take $\nu N = T\mathcal{F}|_N$.

In our earlier paper [2] we discussed differential forms, multivector fields and Riemannian metrics that have a special kind of contact with a normalized submanifold; these were said to be *soldered* to the submanifold. In particular, it was shown that a Riemannian submanifold is totally geodesic iff the metric is soldered to the submanifold and that a submanifold of a Poisson manifold is a (totally) Dirac submanifold iff there exists a normalization such that the Poisson bivector field is soldered to the submanifold with respect to this normalization.

In the present paper we give a general definition for the notion of soldering of an arbitrary tensor field and we consider an obstruction to soldering, which, essentially, is a generalization of the second fundamental form of a Riemannian submanifold. We establish some formulas for the calculation of this obstruction and get corresponding applications. In particular, we prove that the almost complex structure J of an almost Kähler manifold is soldered to a submanifold iff the latter is a J-invariant, totally geodesic submanifold.

2 Soldered tensor fields

Let $(N^n, \nu N)$ be a normalized submanifold of M^m and let $\iota : N \subseteq M$ be the corresponding embedding.

First, we exhibit some adequate, local coordinates around the points of N. Let $\sigma : W \to N$ be a tubular neighborhood of N such that $\forall x \in N$, $T_x(W_x) = \nu_x N$ (W_x is the fiber of W and $\nu_x N$ is the fiber of νN at x). For every point $x \in N$ there exists a σ-trivializing neighborhood U with coordinates (x^a) $(a, b, c, ... = 1, ..., m - n)$ around x on the fibers of σ, such that $x^a|_{N\cap U} = 0$, and coordinates (y^u) $(u, v, w, ... = m - n + 1, ..., m)$ around x on $N \cap U$. We say that (x^a, y^u) are *adapted local coordinates.*

Then,
$$TN|_{N\cap U} = span\left\{ \frac{\partial}{\partial y^u}\Big|_{x^a=0} \right\}, \quad \nu N|_{N\cap U} = span\left\{ \frac{\partial}{\partial x^a}\Big|_{x^a=0} \right\} \tag{2.1}$$
and the transition functions between systems of adapted local coordinates have the local form
$$\tilde{x}^a = \tilde{x}^a(x^b, y^v), \; \tilde{y}^u = \tilde{y}^u(y^v), \tag{2.2}$$
where
$$\frac{\partial \tilde{x}^a}{\partial y^v}\Big|_{x^b=0} = 0, \; \frac{\partial \tilde{y}^u}{\partial x^b} \equiv 0. \tag{2.3}$$
Furthermore, (1.1) implies
$$T^*M|_N = T^*N \oplus \nu^*N, \tag{2.4}$$
and
$$\begin{aligned} T^*N &= ann(\nu N) = span\{dy^u|_{x^a=0}\}, \\ \nu^*N &= ann(TN) = span\{dx^a|_{x^a=0}\} \end{aligned} \tag{2.5}$$
(ann denotes annihilator spaces).

Now we give the following general definition.

Definition 2.1. A tensor field $A \in \mathcal{T}^p_q(M)$ (where $\mathcal{T}$ denotes a space of tensor fields) is *soldered* to the normalized submanifold $(N, \nu N)$ if for any normal vector field $X \in \Gamma TM$ of N one has
$$(L_X A)_x(Y_1, ..., Y_q, \xi_1, ..., \xi_p) = 0, \tag{2.6}$$
for any $x \in N$ and any arguments $Y_1, ..., Y_q \in T_xN, \xi_1, ..., \xi_p \in ann\, \nu_x N$.

In (2.6) L denotes the Lie derivative and it turns out that (2.6) is a combination of algebraic and differential conditions. Indeed, we have

Proposition 2.1. *If the tensor field $A \in \mathcal{T}^p_q(M)$ is soldered to the normalized submanifold $(N, \nu N)$, then, for any fixed vectors $Y_1, ..., Y_q \in T_xN$ and covectors $\xi_1, ..., \xi_p \in ann\, \nu_x N$, the following algebraic conditions must hold:*

1) the 1-forms $\alpha_i \in T^*M|_N$, $i = 1, ..., q$, defined by

$$\alpha_i(V) = A|_N(Y_1, ..., Y_{i-1}, V, Y_{i+1}, ..., Y_q, \xi_1, ..., \xi_p), \quad V \in TM|_N, \tag{2.7}$$

belong to $ann\, \nu N$;

2) the vector fields $Z_j \in TM|_N$, $j = 1, , , , p$, defined by

$$Z_j(\gamma) = A|_N(Y_1, ...Y_q, \xi_1, ..., , \xi_{j-1}, \gamma, \xi_{j+1}, ..., \xi_p), \quad \gamma \in T^*M|_N, \tag{2.8}$$

are tangent to N.

Proof. Consider the general formula

$$(L_{\varphi V}A)(Y_1, ..., Y_q, \xi_1, ..., \xi_p) = \varphi(L_V A)(Y_1, ..., Y_q, \xi_1, ..., \xi_p) \tag{2.9}$$

$$-\sum_{i=1}^{q}(Y_i\varphi)\alpha_i(V) - \sum_{j=1}^{p}\xi_j(V)(Z_j\varphi),$$

where $\alpha_i(V)$ and $Z_j\varphi = d\varphi(Z_j)$ are defined by (2.7), (2.8), respectively; the formula holds for arbitrary arguments (not necessarily related to N) and for any function $\varphi \in C^\infty(M)$. Conditions 1), 2), follow from (2.9) by taking $V = fX + lY$ where $l|_N = 0$; this vector field is normal to N again, therefore, it also satisfies (2.6). □

Proposition 2.2. *The tensor field $A \in \mathcal{T}^p_q(M)$ is soldered to the normalized submanifold N iff the local components of A with respect to adapted coordinates satisfy the conditions*

$$A^{v_1,...,v_p}_{u_1,...,u_{i-1},a,u_{i+1},...,u_q}(0, y^w) = 0, \; A^{v_1,...,v_{j-1},a,v_{j+1},...,v_p}_{u_1,...,u_q}(0, y^w) = 0 \tag{2.10}$$

and

$$\left.\frac{\partial A^{v_1,...,v_p}_{u_1,...,u_q}}{\partial x^a}\right|_{x^b=0} = 0. \tag{2.11}$$

Proof. Using the bases (2.1), (2.5), we see that conditions 1), 2) of Proposition 2.1 are equivalent to (2.10) and (2.11) is (2.6) expressed for $X = \partial/\partial x^a$. Conversely, using formula (2.9), it is easy to derive (2.6) from (2.11) and the algebraic conditions 1), 2). □

In the case of either a differential form or a multivector field formulas (2.10), (2.11) reduce to the conditions for soldering forms and multivector fields given in [2].

Example 2.1. Assume that there exists a foliation $\mathcal{F}$ of M such that $\nu N = T\mathcal{F}|_N$ is a normalization of N. A tensor field $A \in \mathcal{T}^p_q(M)$ is said to be *projectable* or *foliated* if for any local quotient manifold $Q_U = U/\mathcal{F} \cap U$ (U is an open neighborhood in M where $\mathcal{F}$ is simple) there exists a tensor field $A' \in \mathcal{T}^p_q(Q_U)$ that is π-related to A (π is the natural projection $U \to Q_U$). Let (y^u, x^a) be local coordinates such that the local equations of the leaves are $y^u = const$. (In particular, around points $x \in N$ we may use N-adapted local coordinates.) Then, it is easy to see that A is projectable iff it has a local expression of the following form

$$A = dy^{u_1} \otimes ... \otimes dy^{u_q} \otimes [A^{v_1...v_p}_{u_1...u_q}(y)\frac{\partial}{\partial y^{v_1}} \otimes ... \otimes \frac{\partial}{\partial y^{v_p}} \tag{2.12}$$

$$+A^{a_1 v_2 ... v_p}_{u_1...u_q}(x,y)\frac{\partial}{\partial x^{a_1}} \otimes \frac{\partial}{\partial y^{v_2}} ... \otimes \frac{\partial}{\partial y^{v_p}} + ... + A^{a_1 a_2 ... a_p}_{u_1...u_q}(x,y)\frac{\partial}{\partial x^{a_1}} ... \otimes \frac{\partial}{\partial x^{a_p}}].$$

Formula (2.12) shows that the projectable tensor fields are characterized by the following global properties:

(i) $A \in [\otimes^q(ann\, T^*\mathcal{F})] \otimes [\otimes^p TM]$,

(ii) $\forall X \in T\mathcal{F}$ one has $L_X A \in T\mathcal{F} \otimes [\otimes^q(ann\, T^*\mathcal{F})] \otimes [\otimes^{p-1} TM]$.

Accordingly, we see that an $\mathcal{F}$-projectable tensor field A is soldered to the submanifold N iff the algebraic condition 2) of Proposition 2.1 is satisfied. In particular, a totally covariant, foliated tensor field necessarily is soldered to any local transversal submanifold N^n of the foliation $\mathcal{F}^{m-n}$.

Definition 2.2. A tensor field $A \in \mathcal{T}^p_q(M)$ that satisfies the algebraic conditions 1), 2) of Proposition 2.1 (equivalently, satisfies (2.10)) will be called *algebraically adapted to* N.

Proposition 2.3. *If $A \in \mathcal{T}^p_q(M)$ is algebraically adapted to N, the morphism $w_A : \nu N \to \mathcal{T}^p_q(N)$ defined by*

$$w_A(\bar{X})(Y_1, ..., Y_q, \xi_1, ..., \xi_p) = L_X A(Y_1, ..., Y_q, \xi_1, ..., \xi_p)|_N, \tag{2.13}$$

where $Y_1, ..., Y_q \in \Gamma TN$, $\xi_1, ..., \xi_p \in \Gamma(ann\, \nu N)$, $\bar{X} \in \Gamma \nu N$ and X is a vector field on M with the restriction $\bar{X}$ to N, is independent of the choice of the extension X of $\bar{X}$.

Proof. Since A is algebraically N-adapted, formula (2.9) yields

$$(L_{fX+lZ}A)(Y_1, ..., Y_q, \xi_1, ..., \xi_p)|_N = f(L_X A)(Y_1, ..., Y_q, \xi_1, ..., \xi_p)|_N, \tag{2.14}$$

for any functions $f, l \in C^\infty(M)$ such that $l|_N = 0$, any vector field X normal to N and any vector field Z on M. The case $Z = 0$ shows that w_A is $C^\infty(N)$-linear in $\bar{X}$. The case $f = 1$ shows that $w_A(\bar{X})$ is independent of the choice of the extension X of $\bar{X}$, since, using coordinate expressions, it easily follows that two extensions X_1, X_2 are related by an equality of the form $X_2 = X_1 + \sum l_k Z_k$ where the functions l_k vanish on N. □

Definition 2.3. The morphism w_A will be called the *soldering obstruction* of the algebraically N-adapted tensor field A.

The name is motivated by the fact that if $w_A = 0$ then A is soldered to N. Notice that $w_A(\bar{X})$ has the same symmetries like A.

3 Applications

In this section we consider only Riemannian normalizations, therefore, M is endowed with a Riemannian metric g and $\nu N = T^{\perp_g} N$. All the vector fields denoted by Y are tangent to N and all the vector fields denoted by X are normal to N. The reader is asked to pay attention to the situations where calculations take place only along N. In [2], as a consequence of the Gauss-Weingarten formulas [1], we proved that the soldering obstruction of the metric g is

$$w_g(\bar{X})(Y_1, Y_2) = (L_X g)(Y_1, Y_2)|_N = -2g(\beta(Y_1, Y_2), \bar{X}) \tag{3.1}$$

where $X|_N = \bar{X} \in \Gamma\nu N$ and β is the second fundamental form of N. Accordingly, the metric is soldered to a submanifold N iff N is a totally geodesic submanifold of M.

For more applications we compute the soldering obstruction of a tensor field A of type $(1,1)$. From (2.10) it follows that A is algebraically adapted to $(N, \nu N)$ iff both TN and νN are invariant by the endomorphism A and we shall assume that this condition holds. Together with the soldering invariant of A we define the bilinear *soldering form* $\sigma_A(Y_1, Y_2) \in \nu N$ given by

$$g(\sigma_A(Y_1, Y_2), \bar{X}) = w_A(\bar{X})(Y_1, \flat_g Y_2). \tag{3.2}$$

Of course, A is soldered to $(N, T^{\perp_g} N)$ iff $\sigma_A = 0$.

Proposition 3.1. *Assume that the operator A is either symmetric or skew-symmetric with respect to g. Then, σ_A is symmetric, respectively, skew-symmetric iff A is symmetric, respectively, skew-symmetric with respect to the second fundamental form β of the submanifold N of (M, g).*

Proof. The assumed symmetry property is

$$g(AV_1, V_2) = \pm g(V_1, AV_2), \quad \forall V_1, V_2 \in \Gamma TM. \tag{3.3}$$

If we take the Lie derivative L_X of this equality, modulo the equality itself, and use (3.1), we get

$$g(\sigma_A(Y_1, Y_2) \mp \sigma_A(Y_2, Y_1), \bar{X}) = 2g(\beta(AY_1, Y_2) \mp \beta(Y_1, AY_2), \bar{X}),$$

whence the conclusion. □

The following proposition expresses the soldering form of a g-(skew)-symmetric $(1,1)$-tensor field A in terms of the Levi-Civita connection ∇ of g and the second fundamental form β of the submanifold N.

Proposition 3.2. *Assume that the algebraically N-adapted tensor field $A \in \mathcal{T}^1_1(M)$ satisfies* (3.3)*. Then, the following formula, where the sign in the right hand side is opposite to the sign in* (3.3)*, holds:*

$$\begin{aligned} g(\sigma_A(Y_1, Y_2), \bar{X}) &= g(\nabla_{\bar{X}} A(Y_1), Y_2) \\ &+ g(\beta(AY_1, Y_2) \mp \beta(Y_1, AY_2), \bar{X}). \end{aligned} \tag{3.4}$$

Proof. We prove the equality at every fixed point $x \in N$. During the calculations, we extend the vectors $Y_1(x), Y_2(x), \bar{X}(x)$ to vector fields $\tilde{Y}_1, \tilde{Y}_2, X$ on M that are tangent, respectively, normal to N. Since the final result is independent of the choice of the extension, we may use local, adapted coordinates and take

$$\tilde{Y}_1 = \mu_1^u \frac{\partial}{\partial y^u}, \tilde{Y}_2 = \mu_2^u \frac{\partial}{\partial y^u}, X = \xi^a \frac{\partial}{\partial x^a}, \tag{3.5}$$

where μ_1^u, μ_2^u, ξ^a are constant (namely, the components of $Y_1(x), Y_2(x), \bar{X}(x)$ at the fixed point x). From the equality

$$L_X(A\tilde{Y}_1) = [X, A\tilde{Y}_1] = \nabla_X(A\tilde{Y}_1) - \nabla_{A\tilde{Y}_1}X,$$

we get

$$[(L_XA)(Y_1)]_x = [A\nabla_X\tilde{Y}_1 - \nabla_{AY_1}X + (\nabla_XA)(Y_1)]_x.$$

In this result we may replace $\nabla_{A_xY_1(x)}X = -W_{\bar{X}(x)}\tilde{Y}_1 + D_{Y_1(x)}X$, where W is the Weingarten operator of N and D is the connection induced by ∇ in νN. Then, using also (3.3), we get

$$[g(\sigma_A(Y_1, Y_2), \bar{X})]_x = [g(\beta(AY_1, Y_2), \bar{X}) \pm g(\nabla_{\bar{X}}\tilde{Y}_1, AY_2) + g((\nabla_{\bar{X}}A)(Y_1), Y_2)]_x.$$

But, $\nabla_{\bar{X}}\tilde{Y}_1 = \nabla_{Y_1}X + [X, \tilde{Y}_1]$, and the last bracket vanishes for the chosen extensions (3.5). Accordingly,

$$\begin{aligned}[g(\nabla_{\bar{X}}\tilde{Y}_1, AY_2)]_x &= [g(\nabla_{Y_1}X, AY_2) = g(-W_{\bar{X}}Y_1 + D_{Y_1}X, AY_2)]_x \\ &= -[g(\beta_x(Y_1, AY_2), X)]_x\end{aligned}$$

and (3.4) follows. □

Corollary 3.1. *Assume that A is parallel with respect to the Levi-Civita connection of g. Then, if A is g-symmetric $\sigma_A = 0$ and if A is g-skew-symmetric $\sigma_A(Y_1, Y_2) = 2\beta(AY_1, Y_2)$, where β is the Riemannian, second fundamental form of N in M.*

Proof. The Gauss equation

$$\nabla_{Y_1}(A\tilde{Y}_2) = \nabla'_{Y_1}(AY_2) + \beta(AY_1, Y_2),$$

where ∇' is the induced Levi-Civita connection on N, implies that, if $\nabla A = 0$, then $\beta(AY_1, Y_2) = A\beta(Y_1, Y_2)$. Similarly, $\beta(Y_1, AY_2) = A\beta(Y_1, Y_2)$. Inserting $\nabla A = 0$ and the previous results for β in (3.4) we get the announced results. □

Another nice formula is given by

Proposition 3.3. *Let A be either a g-symmetric or a g-skew-symmetric $(1,1)$-tensor field on (M, g) that is algebraically adapted to the submanifold N and let $\mathcal{N}_A$ be its Nijenhuis tensor. Then, one has*

$$g(\sigma_A(Y_1, Y_2), A\bar{X}) = \pm g(\sigma_A(Y_1, AY_2), \bar{X}) + g(\mathcal{N}_A(X, Y_1), Y_2). \tag{3.6}$$

Proof. With the notation in the proof of Proposition 3.2, if $x \in N \subseteq M$, one has the following expression of the Nijenhuis tensor

$$[(\mathcal{N}_A)(X,\tilde{Y}_1)]_x = [(L_{AX}A)(\tilde{Y}_1) - A(L_X A)(\tilde{Y}_1)]_x. \tag{3.7}$$

The required result follows by taking the g-scalar product of the previous equality by Y_2. □

For a concrete application, let (M, J, g) be an almost Hermitian manifold, which means that the almost complex structure J is g-skew-symmetric. The tensor field J is algebraically adapted to the submanifold N iff N is J-invariant, which we shall assume hereafter. If (M, J, g) is a Kähler manifold and N is a complex submanifold, Corollary 3.2 gives $\sigma_J(Y_1, Y_2) = 2\beta(JY_1, Y_2)$ and we see that J is soldered to $(N, T^{\perp_g}N)$ iff N is a totally geodesic submanifold.

We shall extend this result to almost Kähler manifolds. For any almost Hermitian manifold (M, J, g) one has the Kähler form $\Omega(Y_1, Y_2) = g(JY_1, Y_2)$.

Proposition 3.4. *The soldering form of the almost complex structure J is related to the Kähler form Ω by means of the formula*

$$g(\sigma_J(Y_1, Y_2), \bar{X}) = 2g(\beta(JY_1, Y_2), \bar{X}) + d\Omega(\bar{X}, Y_1, Y_2). \tag{3.8}$$

Proof. From the definition of Ω we get

$$(L_X\Omega)(Y_1, Y_2) = (L_X g)(JY_1, Y_2) + g(L_X J(Y_1), Y_2),$$

which, for $\bar{X} \in \nu N$ for $Y_1, Y_2 \in TN$ becomes

$$g(\sigma_J(Y_1, Y_2), \bar{X}) = (L_X\Omega)(Y_1, Y_2) - g(\sigma_g(JY_1, Y_2), \bar{X}). \tag{3.9}$$

Notice that Ω is algebraically compatible with $(N, T^{\perp_g})$, therefore,

$$(L_X\Omega)(Y_1, Y_2) = g(\sigma_\Omega(Y_1, Y_2), \bar{X}).$$

Since it is easy to check that for the involved arguments one has $(di(X)\Omega)(Y_1, Y_2) = 0$, by using $L_X = di(X) + i(X)d$ in (3.9), we get

$$g(\sigma_J(Y_1, Y_2), \bar{X}) = d\Omega(\bar{X}, Y_1, Y_2) - g(\sigma_g(JY_1, Y_2), \bar{X}). \tag{3.10}$$

In view of (3.1), formula (3.10) is the same as the one required by the proposition. □

Corollary 3.2. *The almost complex structure J is soldered to the J-invariant submanifold N iff the second fundamental form of N is given by the formula*

$$g(\beta(JY_1, Y_2), X) = -\frac{1}{2}d\Omega(X, Y_1, Y_2). \tag{3.11}$$

Then, since an almost Kähler manifold is characterized by the property $d\Omega = 0$, we get the main application:

Proposition 3.5. *If (M, J, g) is an almost Kähler manifold, the almost complex structure J is soldered to the submanifold N iff N is a J-invariant, totally geodesic submanifold of M.*

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