LIBERTAS MATHEMATICA, vol XXX (2010)

L_p-Width-Integrals and Affine Surface Areas

Chang-jian ZHAO and Mihály BENCZE

Abstract. The main purposes of this paper are to establish some new Brunn-Minkowski inequalities for L_p-width-integrals of mixed projection bodies and L_p-affine surface area of mixed bodies.

Keywords: L_p-width-integrals, L_p-affine surface area, mixed projection body, mixed body.

Mathematics Subject Classification (2000): 52A40, 53A15, 46B20.

§1. Introduction

In recent years some authors including Ball[1], Bourgain[2], Gardner[3], Schneider[4] and Lutwak[5−10] et al have given considerable attention to the Brunn-Minkowski theory and Brunn-Minkowski-Firey theory and their various generalizations. In particular, Lutwak[7] had generalized the Brunn-Minkowski inequality (1) to mixed projection body and get inequality (2):

The Brunn-Minkowski inequality *If $K, L \in \mathcal{K}^n$, then*

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1}$$

with equality if and only if K and L are homothetic.

The Brunn-Minkowski inequality for mixed projection bodies *If $K, L \in \mathcal{K}^n$, then*

$$V(\Pi(K+L))^{1/n(n-1)} \geq V(\Pi K)^{1/n(n-1)} + V(\Pi L)^{1/n(n-1)}, \tag{2}$$

with equality if and only if K and L are homothetic.

On the other hand, width-integral of convex bodies and affine surface areas play an important role in the Brunn-Minkowski theory. Width-integrals were first considered by Blaschke[11] and later by Hadwiger[12]. In addition, Lutwak had established the following results for the width-integrals of convex bodies and affine surface areas.

Supported by National Natural Sciences Foundation of China (10971205).

The Brunn-Minkowski inequality for width-integrals of convex bodies[10]

If $K, L \in \mathcal{K}^n$, $i < n-1$

$$B_i(K+L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)} \tag{3}$$

with equality if and only if K *and* L *have similar width.*

The Brunn-Minkowski inequality for affine surface area[9]

If $K, L \in \kappa^n$, *and* $i \in \mathbb{R}$, *then for* $i < -1$

$$\Omega_i(K\tilde{+}L)^{(n+1)/(n-i)} \leq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)} \tag{4}$$

with equality if and only if K *and* L *are homothetic , while for* $i > -1$

$$\Omega_i(K\tilde{+}L)^{(n+1)/(n-i)} \geq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)} \tag{5}$$

with equality if and only if K *and* L *are homothetic.*

In this paper, we firstly generalize inequality (3) to the L_p-width-integrals of mixed projection bodies and get the following result.

Result A *If* $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$, *and* $C = (K_3, \ldots, K_n)$, *then for* $p \geq 1$, $i < n-1$

$$B_{p,i}(\Pi(C, K_1+K_2))^{1/(n-i)} \leq B_{p,i}(\Pi(C,K_1))^{1/(n-i)} + B_{p,i}(\Pi(C,K_2)^{1/(n-i)}, \tag{6}$$

with equality if and only if $\Pi(C, K_1)$ *and* $\Pi(C, K_2)$ *are homothetic.*

Secondly, we prove that analogs of inequalities (4)-(5) for L_p-affine surface area of mixed bodies.

Result B *If* $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ *and all of mixed bodies of* $K_1, K_2, \ldots, K_n$ *have positive continuous curvature functions, then for* $p \geq 1$

$$\Omega_p([K_1+K_2, K_3, \ldots, K_n])^{(n+p)/n}$$

$$\geq \Omega_p([K_1, K_3, K_4 \ldots, K_n])^{(n+p)/n} + \Omega_p([K_2, K_3, \ldots, K_n])^{(n+p)/n} \tag{7}$$

with equality if and only if $[K_1, K_3, K_4, \ldots, K_n]$ *and* $[K_2, K_3 \ldots, K_n]$ *are homothetic.*

Please see the next section for above interrelated notations, definitions and their background materials.

§2. Notations and Preliminary works

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n(n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures(compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$, and if $p \in \mathcal{K}^n$, let $\mathcal{K}_p^n$ denote the subset of $\mathcal{K}^n$ that contains the centered (centrally symmetric with respect to p) bodies. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u.

2.1 Mixed volumes

We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K, \cdot) : S^{n-1} \to \mathbb{R}$, denote the support function of $K \in \mathcal{K}^n$; i.e.

$$h(K, u) = Max\{u \cdot x : x \in K\}, u \in S^{n-1}, \tag{8}$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

For a convex body K and a nonnegative scalar $\lambda, \lambda K$, is used to denote $\{\lambda x : x \in K\}$. For $K_i \in \mathcal{K}^n, \lambda_i \geq 0, (i = 1, 2, \ldots, r)$,the Minkowski linear combination $\sum_{i=1}^r \lambda_i K_i \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in K^n : x_i \in K_i\}. \tag{9}$$

It is trivial to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_r K_r, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_r h(K_r, \cdot). \tag{10}$$

If $K_i \in \mathcal{K}^n(i = 1, 2, \ldots, r)$ and $\lambda_i(i = 1, 2, \ldots, r)$are nonnegative real numbers, then of fundamental impotence is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by [4]

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{11}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (11), it is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1 \ldots K_n)$ is usually written $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the ith projection measure(Quermassintegral) of K and is written as $W_i(K)$. With this notation, $W_0 = V(K)$, while $nW_1(K)$ is the surface area of K, $S(K)$.

2.2 L_p-Width-integrals of convex bodies

For $u \in S^{n-1}$, $b(K, u)$ is defined to be half the width of K in the direction u. Two convex bodies K and L are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$ for all $u \in S^{n-1}$. For $K \in \mathcal{K}^n$ and $p \in intK$, we use K^p to denote the polar reciprocal of K with respect to the unit sphere centered at p. The width-integral of index i is defined by Lutwak[10]: For $K \in \mathcal{K}^n, i \in \mathbb{R}$

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u),$$

where dS is the $(n-1)$-dimensional volume element on S^{n-1}.

The width-integral of index i is a map

$$B_i : \mathcal{K}^n \to \mathbb{R}.$$

It is positive, continuous, homogeneous of degree $n - i$ and invariant under motion. In addition, for $i \leq n$ it is also bounded and monotone under set inclusion.

L_p width-integral of index i is defined by

$$B_{p,i}(K) = \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} b(K, u)^{p(n-i)} dS(u) \right)^{1/p}, \quad p \geq 1. \tag{12}$$

The following result[10] will be used later

$$b(K + L, u) = b(K, u) + b(L, u), \tag{13}$$

2.3 The radial function and the Blaschke linear combination

The radial function of convex body K, $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K,\cdot) = Max\{\lambda \geq 0 : \lambda\mu \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be call a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$.

A convex body K is said to have a positive continuous curvature function[5],

$$f(K,\cdot) : S^{n-1} \to [0,\infty),$$

if for each $L \in \varphi^n$, the mixed volume $V_1(K,L)$ has the integral representation

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} f(K,u)h(L,u)dS(u).$$

The subset of $\mathcal{K}^n$ consisting of bodies which have a positive continuous curvature function will be denoted by κ^n. Let κ^n_c denote the set of centrally symmetric member of κ^n.

The following result is true[6], for $K \in \kappa^n$

$$\int_{S^{n-1}} uf(K,u)dS(u) = 0.$$

Suppose $K, L \in \kappa^n$ and $\lambda, \mu \geq 0$(not both zero). From above it follows that the function $\lambda f(K,\cdot) + \mu f(L,\cdot)$ satisfies the hypothesis of Minkowski's existence theorem(see [13]). The solution of the Minkowski problem for this function is denoted by $\lambda \cdot K \tilde{+} \mu \cdot L$ that is

$$f(\lambda \cdot K \tilde{+} \mu \cdot L, \cdot) = \lambda f(K,\cdot) + \mu f(L,\cdot), \tag{14}$$

where the linear combination $\lambda \cdot K \tilde{+} \mu \cdot L$ is called a Blaschke linea combination.

Similarly, for L_p-curvature function, we have

$$f_p(\lambda \cdot K \tilde{+} \mu \cdot L, \cdot) = \lambda f_p(K,\cdot) + \mu f_p(L,\cdot), \tag{15}$$

2.4 Mixed affine area and mixed bodies

The affine surface area of $K \in \kappa^n$, $\Omega(K)$, is defined by

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{n/(n+1)} dS(u). \tag{16}$$

It is well known that this functional is invariant under unimodular affine transformations. For $K, L \in \kappa^n$, and $i \in \mathbb{R}$, the ith mixed affine surface area of K and L, $\Omega_i(K, L)$, was defined in[5] by

$$\Omega_i(K, L) = \int_{S^{n-1}} f(K, u)^{(n-i)/(n+1)} f(L, u)^{i/(n+1)} dS(u). \tag{17}$$

Now, we define the ith affine area of $K \in \kappa^n, \Omega_i(K)$, to be $\Omega_i(K, B)$, since $f(B, \cdot) = 1$ one has

$$\Omega_i(K) = \int_{S^{n-1}} f(K, u)^{(n-i)/(n+1)} dS(u), \quad i \in \mathbb{R}.$$

the Lp-affine surface area, $\Omega_p(K)$, of convex body K is defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{n/(n+p)} dS(u), \tag{18}$$

Lutwak[8] defined mixed bodies of convex bodies $K_1, \ldots, K_{n-1}$ as $[K_1, \ldots, K_{n-1}]$. The following property will be used later:

$$[K_1 + K_2, K_3, \ldots, K_n] = [K_1, K_3, \ldots, K_n] \tilde{+} [K_2, K_3, \ldots, K_n] \tag{19}$$

2.5 Mixed projection bodies

If $K_i (i = 1, 2, \ldots, n-1) \in K^n$, then the mixed projection body of K_i $(i = 1, 2, \ldots, n-1)$ is denoted by $\Pi(K_1, \ldots, K_{n-1})$, and whose support function is given, for $u \in S^{n-1}$, by[7]

$$h\left(\Pi(K_1, \ldots, K_{n-1}), u\right) = v(K_1^u, \ldots, K_{n-1}^u). \tag{20}$$

It is easy to see, $\Pi(K_1, \ldots, K_{n-1})$ is centered.

If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_1, \ldots, K_{n-1})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, B)$ is called the ith projection body of K and is denoted $\Pi_i K$. We write $\Pi_0 K$ as ΠK.

The following property will be used:

$$\Pi(K_3, \ldots, K_n, K_1 + K_2) = \Pi(K_3, \ldots, K_n, K_1) + \Pi(K_3, \ldots, K_n, K_2) \tag{21}$$

§3. Main results and their proofs

Our main results are the following results which were stated in the introduction.

Theorem 1 *If* $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ *and* $C = (K_3, \ldots, K_n)$, *then for* $p \geq 1,\ i < n-1$

$$B_{p,i}(\Pi(C, K_1+K_2))^{1/(n-i)} \leq B_{p,i}(\Pi(C, K_1))^{1/(n-i)} + B_{p,i}(\Pi(C, K_2)^{1/(n-i)}, \quad (22)$$

with equality if and only if $\Pi(C, K_1)$ *and* $\Pi(C, K_2)$ *are homothetic.*

Proof From (12), (13), (21) and notice for $i < n-1$ to use the Minkowski inequality for integral[14,P.147], we obtain for $p \geq 1$

$$B_{p,i}(\Pi(C, K_1 + K_2))^{1/(n-i)} =$$

$$= \left(\omega_n \left(\frac{1}{n\omega_n}\int_{S^{n-1}} b(\Pi(C, K_1 + K_2), u)^{p(n-i)} dS(u)\right)^{1/p}\right)^{n-i}$$

$$= \left(\omega_n \left(\frac{1}{n\omega_n}\int_{S^{n-1}} b(\Pi(C, K_1) + \Pi(C, K_2), u)^{p(n-i)} dS(u)\right)^{1/p}\right)^{n-i}$$

$$= \left(\omega_n \left(\frac{1}{n\omega_n}\int_{S^{n-1}} (b(\Pi(C, K_1), u) + b(\Pi(C, K_2), u))^{p(n-i)} dS(u)\right)^{1/p}\right)^{n-i}$$

$$\leq \left(\omega_n \left(\frac{1}{n\omega_n}\int_{S^{n-1}} b(\Pi(C, K_1), u)^{p(n-i)} dS(u)\right)^{1/p}\right)^{n-i}$$

$$+ \left(\omega_n \left(\frac{1}{n\omega_n}\int_{S^{n-1}} b(\Pi(C, K_2), u)^{p(n-i)} dS(u)\right)^{1/p}\right)^{n-i}$$

$$= B_{p,i}(\Pi(C, K_1))^{1/(n-i)} + B_{p,i}(\Pi(C, K_2))^{1/(n-i)},$$

with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ have similar width, in view of $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are centered (centrally symmetric with respect to origin),then with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic. The proof of inequality (22) is complete.

Taking $p = 1$ to (22), (22) changes to the following result

Corollary 1 *If* $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ *and* $C = (K_3, \ldots, K_n)$, *then*

$$B_i(\Pi(C, K_1 + K_2))^{1/(n-i)} \leq B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2)^{1/(n-i)}, \quad (23)$$

with equality if and only if $\Pi(C, K_1)$ *and* $\Pi(C, K_2)$ *are homothetic.*

Taking $p = 1, i = 0$ to (22), inequality (22) changes to the following result

Corollary 2 *If $K_1, K_2, \dots, K_n \in \mathcal{K}^n$, let $C = (K_3, \dots, K_n)$, then*

$$B(\Pi(C, K_1 + K_2))^{1/n} \leq B(\Pi(C, K_1))^{1/n} + B(\Pi(C, K_2)^{1/n}, \tag{24}$$

with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.

Theorem 2 *If $K_1, K_2, \dots, K_n \in \mathcal{K}^n$ and all of mixed bodies of $K_1, K_2, \dots, K_n$ have positive continuous curvature functions, then for $p \geq 1$*

$$\Omega_p([K_1 + K_2, K_3, \dots, K_n])^{(n+p)/n}$$

$$\geq \Omega_p([K_1, K_3, K_4 \dots, K_n])^{(n+p)/n} + \Omega_p([K_2, K_3, \dots, K_n])^{(n+p)/n} \tag{25}$$

with equality if and only if $[K_1, K_3, K_4, \dots, K_n]$ and $[K_2, K_3 \dots, K_n]$ are homothetic.

Proof From (15), (18), (19) and in view of the Minkowski inequality for integral[14,P.147], we obtain that

$$\Omega_p([K_1 + K_2, K_3, K_4, \dots, K_n])^{(n+p)/n}$$

$$= \left(\int_{S^{n-1}} f_p([K_1 + K_2, K_3, K_4, \dots, K_n], u)^{n/(n+p)} dS(u) \right)^{(n+p)/n}$$

$$= \left(\int_{S^{n-1}} f_p([K_1, K_3, K_4, \dots, K_n] \tilde{+} [K_2, K_3, \dots, K_n], u)^{n/(n+p)} dS(u) \right)^{(n+p)/(n}$$

$$= \left(\int_{S^{n-1}} (f_p([K_1, K_3, K_4, \dots, K_n], u) + f_p([K_2, K_3, \dots, K_n], u))^{n/(n+p)} dS(u) \right)^{(n+p)/n}$$

$$\geq \left(\int_{S^{n-1}} f_p([K_1, K_3, K_4, \dots, K_n], u)^{n/(n+p)} dS(u) \right)^{(n+p)/n}$$

$$+ \left(\int_{S^{n-1}} f_p([K_2, K_3, \dots, K_n], u)^{n/(n+p)} dS(u) \right)^{(n+p)/n}$$

$$= \Omega_p([K_1, K_3, K_4, \dots, K_n])^{(n+p)/n} + \Omega_p([K_2, K_3, \dots, K_n])^{(n+p)/n},$$

with equality if and only if $[K_1, K_3, K_4, \dots, K_n]$ and $[K_2, K_3, \dots, K_n]$ are homothetic.

The proof of Theorem 2 is complete.

Taking $p = 1$ to (25), we have

Corollary 3 *If $K_1, K_2, \dots, K_n \in \mathcal{K}^n$ and all of mixed bodies of $K_1, K_2, \dots, K_n$ have positive continuous curvature functions, then*

$$\Omega([K_1 + K_2, K_3, \dots, K_n])^{(n+1)/n}$$
$$\geq \Omega([K_1, K_3, K_4, \dots, K_n])^{(n+1)/n} + \Omega([K_2, K_3, \dots, K_n])^{(n+1)/n},$$

with equality if and only if $[K_1, K_3, K_4, \dots, K_n]$ and $[K_2, K_3 \dots, K_n]$ are homothetic.

References

[1] K. Ball, "Volume of sections of cubes and related problems", Israel Seminar (G.A.F.A.) 1988, Lecture Notes in Math. Vol.1376, Springer–Verlag, Berlin and New York, 1989, 251–260.

[2] J.Bourgain and J.Lindenstrauss, "Projection bodies", Israel Seminar (G.A.F.A) 1986–1987, Lecture Notes in Math. Vol.**1317**, Springer-Verlag, Berlin and New York, 1988, 250–270.

[3] R. J. Gardner, "Geometric Tomography", Cambridge: Cambridge University Press, 1995.

[4] R. Schneider, "Convex bodies: The Brunn-Minkowski Theory", Cambridge: Cambridge University Press, 1993.

[5] E. Lutwak, Centroid bodies and dual mixed volumes, *Proc. London Math. Soc.*, **60**(1990), 365–391.

[6] E. Lutwak, Mixed projection inequalities, *Trans. Amer. Math. Soc,* **287**(1985), 92–106.

[7] E. Lutwak, Inequalities for mixed projection, *Tans. Amer. Math. Soc.*, **339**1993, 901-916.

[8] E. Lutwak, Volume of mixed bodies, *Trans. Amer. Math. Soc.*, **294**(1986), 487-500.

[9] E. Lutwak, Mixed affine surface area, *J. Math. Anal. Appl.*, **125**(1987), 351-360.

[10] E. Lutwak, Width-integrals of convex bodies, *Proc. Amer. Math. Soc.*, **53**(1975), 435-439.

[11] W. Blaschke, "Vorlesungen *ü*ber Integral geometric I, II", Teubner, Leipzig, 1936, 1937; reprint, chelsea, New York, 1949.

[12] Hadwiger H, "Vorlesungen *ü*ber inhalt, Oberfl*ä*che und isoperimetrie, Springer, Berlin, 1957.

[13] T. Bonnesen and W, Fenchel, "Theorie der konvexen K*ö*rper", Springer, Berlin, 1934.

[14] G. H. Hardy, J. E. Littlewood and G. P*ólya*, "Inequalities", Cambridge Univ. Press. Cambridge, 1934.