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Green Function of the Laplacian for the Neumann Problem in $\mathbb{R}^n_+$

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Abstract. By analogy with the Green function for the Dirichlet problem for a domain Ω, one considers the Green type function for the Neumann problem for Ω. We obtain the Green type function for the positive half-space of $\mathbb{R}^n$ and use it to solve the Neumann problem. The cases where Ω is the positive half space of $\mathbb{R}^2$ and of $\mathbb{R}^3$ are analyzed in details under new hypotheses.

Keywords: Green's function for the Neumann problem, Neumann function, Dirichlet problem.

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1 Introduction

Let Ω be a domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Recall the Dirichlet and Neumann problems for the Laplacian.

(1) Dirichlet Problem (DP):

Find $u \in C^2(\Omega) \cap C(\bar{\Omega}))$ such that

$$\Delta u = f \text{ in } \Omega,\ u = g \text{ on } \partial\Omega. \tag{1}$$

(2) Neumann Problem (NP):

Find $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\Delta u = f \text{ in } \Omega, \quad \frac{du}{d\eta_x} = g_1 \text{ on } \partial\Omega, \tag{2}$$

where η_x is the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$, $\dfrac{du}{d\eta_x}$ is the normal derivative of u at x, $f : \Omega \to \mathbb{R}$, and $g,\ g_1 : \partial\Omega \to \mathbb{R}$ are prescribed continuous functions.

By analogy with the Green function for the Dirichlet problem for a domain Ω, we consider the Green type function for the Neumann problem for Ω (also known as Neumann's function, or Green's function for the Neumann problem or Green's function of the second kind). We will give the explicit forms of the Neumann's functions and the solution of the Neumann problem for the upper half-space of the n-th dimensional euclidian space $\mathbb{R}^n$, $n \geq 2$.

The concept of Green type function for (NP) has been considered by several authors ([14], [9], [19], [10], [17], [15], [2], [6], [12], [13]).

In [3], there are presented the expressions of the Green's functions for the Neumann's problem for a ball in $\mathbb{R}^2$ and $\mathbb{R}^3$. In [12], Neumann's function for the sphere in $\mathbb{R}^3$ is constructed using the classical method of images and expressed in terms of eigenvalues associated with the surface, leading to an analogue of the Poisson integral as a solution to the Neumann problem for the sphere. In [6], there are given the Neumann's function and the solution of the Neumann problem for the interior and the exterior of the sphere of $\mathbb{R}^n$, $n \geq 2$.

Green functions of the Laplacian for Neumann problems relative to all domains bounded by the coordinate surfaces in the circular cylindrical coordinate system are constructed in [2].

In [13] it can be found a construction of the Green function for the three-dimensional Laplace equation, in the interior of an arbitrary rectangular channel subject to homogeneous Neumann conditions on the boundaries.

The explicit forms of the Green's function and of the Neumann's function G and G_1 in the half-plane $x_2 \geq 0$ in $\mathbb{R}^2$ are given in [10], while in [17] there are given the Green's and Neumann's functions for both the half-plane $x_2 \geq 0$ and the upper half-space $x_3 \geq 0$ in $\mathbb{R}^3$. However, in both books it is not proved that the functions u and u_1 obtained by means of G and G_1 and Green's formula are solutions of the (DP) and (NP), respectively.

In [9], the explicit expressions of the Green's and the Neumann's functions are given for the upper half-spaces $\mathbb{R}^2_+$ and $\mathbb{R}^3_+$, and it is shown that the functions u and u_1 obtained with their aid are solutions of the (DP) and (NP), respectively, under the hypothesis that g and g_1 are analytical. Physical interpretations of the Green's and Neumann's functions can be found in [9]. Also the Green's and Neumann's functions for the interior and the exterior of the unit circle in $\mathbb{R}^2$ and unit sphere in $\mathbb{R}^3$ centered at the origin are given in [9].

In [19], it is constructed the Green's function for the Neumann problem formulated for the unit sphere in $\mathbb{R}^3$ centered at the origin. Also in [19], there are obtained the explicit solutions u and u_1 of the (DP) and (NP) using Green's formula for the tridimensional upper-half space, respectively (without using the Green's function for (DP) or the Green's function for (NP)). Then it is proved that u and u_1 are indeed solutions of (DP) and (NP), respectively, under the assumptions: g and g_1 are continuous on $x_3 = 0$, g is bounded and $|g(x_1, x_2)| \leq \frac{M}{(\sqrt{x_1^2+x_2^2})^{1+a}}$, where $0 < a < 1$, and M is a positive constant.

In [4], it is derived the Green's function and it is shown a Poisson Formula for the positive half-space of $\mathbb{R}^n$, $n \geq 2$.

We obtain the Green type function for the positive half-space $\mathbb{R}^n_+$, $n \geq 2$, and use it to find the solution of the (NP) under the assumption that g is continuous and bounded on $\mathbb{R}^{n-1}$, and g_1 is continuous and with compact support in $\mathbb{R}^{n-1}$ (subsection 3.3). The cases where Ω is the positive half space $\mathbb{R}^2_+$ and $\mathbb{R}^3_+$, respectively, are presented in details under hypotheses different from the ones considered by other authors (subsection 3.1 and subsection 3.2).

2 Green Functions for Dirichlet and Neumann Problems

Suppose that the function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then the following (Riemann-Green) formula holds, i.e.,

$$u(y) = \int_\Omega G(x,y)\Delta u(x)\,dx - \int_{\partial\Omega} (G(x,y)\frac{du}{d\eta_x}(x) - u(x)\frac{dG(x,y)}{d\eta_x})\,d\sigma_x, \tag{3}$$

where $G(x,y) = \psi(\|x-y\|) + \tilde{g}(x),\ x \in \bar{\Omega},\ y \in \Omega,\ x \neq y,\ \tilde{g}(x)$ is an arbitrary harmonic function in Ω and

$$\psi(r) = \frac{r^{2-n}}{(2-n)\sigma_n}, \text{ if } n > 2 \text{ and } \psi(r) = \frac{1}{2\pi}\ln r, \text{ if } n = 2, \tag{3'}$$

with $r = \|x-y\|$. Finally, σ_n is the surface area of the unit sphere in $\mathbb{R}^n$.
Suppose that the function $G(x,y)$ in the above formula satisfies the additional property

$$G(x,y) = 0 \text{ for } x \in \partial\Omega,\ y \in \Omega. \tag{4}$$

Then the solution u of (DP) with the regularity $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ (if any) is given by

$$u(y) = \int_\Omega G(x,y)f(x)dx + \int_{\partial\Omega} g(x)\frac{dG(x,y)}{d\eta_x}d\sigma_x. \tag{5}$$

A Green's function $G(x,y)$ for Δ on Ω is a function G as above, i.e., having the properties $x \to G(x,y)$ belongs to $C^2(\bar{\Omega} \setminus \{y\})$, $\Delta_x G(x,y) = 0$ for $x \in \Omega$, $G(x,y) = 0$, for $x \in \partial\Omega$, $y \in \Omega$.

For the Neumann problem, u is not prescribed on the boundary $\partial\Omega$ of Ω, so the formula (3) suggests to look for a function $G = G_1$ with the condition

$$\frac{dG_1(x,y)}{d\eta_x} = 0, \text{ for } x \in \partial\Omega, \tag{6}$$

instead of $G(x,y) = 0$, for $x \in \partial\Omega$ for the Dirichlet problem. This means to find a function $\tilde{g}(x) = K(x,y)$ with $x \to K(x,y)$ in $C^1(\bar{\Omega}) \cap C^2(\Omega)$, $\forall\, y \in \Omega$,

$$\Delta_x K(x,y) = 0,\ \forall\, x,\ y \in \Omega,$$

$$\frac{dK(x,y)}{d\eta_x} = -\frac{d\psi\|x-y\|}{d\eta_x} = -\psi'(r)\frac{x-y}{r} \cdot \eta_x,\ x \in \partial\Omega,\ y \in \Omega,$$

where ψ is given above, i.e., $\psi'(r) = \dfrac{r^{1-n}}{\sigma_n}$. Therefore, the solution $u_1 \in C^2(\Omega) \cap C^1(\bar{\Omega})$ (if any) of the (NP) is necessarily given by

$$u_1(y) = \int_\Omega G_1(x,y)f(x)\,dx - \int_{\partial\Omega} G_1(x,y)g_1(x)d\sigma_x. \tag{7}$$

Such a function G_1 will be called a Green type function for the (NP) on Ω (also called Green's function for (NP) in [14], [19], [3], [2], or Neumann's function in [9], [17], [15], or Green's function of the second kind in [10]).

3 Green function for the Neumann Problem for $\mathbb{R}^n_+$

In this section we will build Green's functions for the (DP) and (NP) for the half-space $\mathbb{R}^n_+$, $n \geq 2$, using the ideas developed in Section 2. After that, we will check directly that the corresponding representation formulas for the solutions of (DP) and (NP) are valid under appropriate assumptions on g and g_1.

First we consider the cases $n = 2$ and $n = 3$ under hypothesis on g_1 different from the ones used by other authors.

3.1 Construction of the Green function for the Neumann Problem for $\mathbb{R}^2_+$

[14] Let $\Omega = \{x = (x_1, x_2) \in R^2, x_2 > 0\}$ be the positive half-space. Clearly, $\partial\Omega = \{x = (x_1, x_2) \in R^2, x_2 = 0\} = \{x = (x_1, 0), x_1 \in R\}$. For $y = (y_1, y_2)$ define by reflection $y^* = (y_1, -y_2)$. Then the function:

$$G(x, y) = \frac{1}{2\pi}(ln\|x - y\| - ln\|x - y^*\|), \quad x \in \bar{\Omega},\ y \in \Omega,\ x \neq y, \tag{8}$$

is a Green function for the (DP), and

$$G_1(x, y) = \frac{1}{2\pi}(ln\|x - y\| + ln\|x - y^*\|), \quad x \in \bar{\Omega},\ y \in \Omega,\ x \neq y, \tag{9}$$

is a Green type function for (NP).

As $\|x-y\| = \|x-y^*\| = r$ for $y \in \Omega,\ x \in \partial\Omega$, it follows that $G(x, y) = 0$, $x \in \partial\,\Omega$, $y \in \Omega$.

The outward normal η_x to $\partial\Omega = \{x = (x_1, x_2) \in R^2, x_2 = 0\}$ at $x \in \partial\Omega$ is $\eta_x = (0, -1)$. The normal derivative of G is:

$$\frac{dG}{d\eta_x} = -\frac{\partial G}{\partial x_2} = \pi^{-1} y_2 r^{-2} \text{ with } x = (x_1, 0), y = (y_1, y_2), y_2 > 0,$$

where $r^2 = (x_1 - y_1)^2 + y_2^2$. Similarly we can check that $\dfrac{dG_1}{d\eta_x} = 0$.

The formula (3) suggests that a solution to the problem: $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, could be

$$u(y) = \int_{\partial\Omega} u(x)\frac{dG}{d\eta_x}(x, y)\, d\sigma_x = \pi^{-1}\int_{-\infty}^{\infty} \frac{y_2 g(x_1)dx_1}{(x_1 - y_1)^2 + y_2^2}, \tag{10}$$

with $x = (x_1, 0), g(x_1, 0) = g(x_1), y = (y_1, y_2),\ y_2 > 0$.

Clearly $G_1(x, y) = \dfrac{1}{\pi} ln\|x - y\|$, for $x = (x_1, 0),\ y \in \Omega$. Therefore, (3) suggests that a possible solution to the Neumann problem (with $f = 0$) could be

$$u_1(y) = -\int_{\partial\Omega} G_1(x, y) g_1(x) d\sigma_x = -\pi^{-1}\int_{-\infty}^{\infty} g_1(x_1) ln\|x - y\| dx_1, \tag{11}$$

with $x = (x_1, 0),\ g_1(x_1, 0) = g_1(x_1),\ y = (y_1, y_2),\ \ y_2 > 0.$

The functions g and g_1 must guarantee the convergence of the improper integrals (10) and (11), respectively (i.e., the existence of u and u_1), and the fact that these functions u and u_1 are solutions of the above Dirichlet and Neumann problems with $f = 0$. An important case in which these requirements are fulfilled is given by:

Theorem 1 *Let g be continuous and bounded on $\mathbb{R}$ and g_1 be continuous with compact support in $\mathbb{R}$. Then the functions u and u_1 given by the improper integrals (10) and (11) satisfy:*
1. $u \in C^2(\Omega) \cap C(\bar{\Omega})), \Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$.
2. $u_1 \in C^2(\Omega) \cap C^1(\bar{\Omega}), \Delta u_1 = 0$ in Ω, $\dfrac{du_1}{d\eta_x} = g_1$, on $\partial\Omega$.

Proof. The key fact is the elementary formula:

$$\pi^{-1} \int_{-\infty}^{\infty} \frac{y_2 dx_1}{(x_1 - y_1)^2 + y_2^2} = 1,\ x = (x_1, 0), y = (y_1, y_2),\ y_2 > 0. \tag{12}$$

which follows by the change of variable: $x_1 - y_1 = y_2 z$, so $dx_1 = y_2 dz$.

The fact that u and u_1 are harmonic in Ω follows by differentiating under the integral sign, in conjunction with:
$\Delta_y \dfrac{dG}{d\eta_x}(x, y) = \dfrac{d}{d\eta_x}(\Delta_y G(x, y)),\ \ G(x, y) = G(y, x)$ for $x \in \bar{\Omega},\ y \in \Omega,\ x \neq y$, and $\Delta_y G(x, y) = \Delta_y G(y, x) = 0$ for $x \in \partial\Omega,\ y \in \Omega$.

As g is bounded and g_1 is continuous and with compact support in $\mathbb{R}$, u and u_1 given by (10) and (11) are bounded.

Let us to prove that $u(x) = g(x)$ for $x = (x_1, 0)$, i.e., that

$$\lim_{y \to y_0} u(y) = g(y_0),\ \forall\, y_0 \text{ with } \ y_0 = (y_0^1, 0), y = (y_1, y_2), \text{with } y_2 > 0. \tag{13}$$

In view of (10), we can write

$$u(y) - g(y_0) = \pi^{-1} \int_{-\infty}^{\infty} \frac{y_2(g(x_1) - g(y_0^1))dx_1}{(x_1 - y_1)^2 + y_2^2}, \tag{14}$$

$y = (y_1, y_2),\ y_2 > 0.$

Given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$|g(x_1) - g(y_0^1)| < \epsilon, \text{ for } |x_1 - y_0^1| < \delta.$$

Let us write:

$$u(y) - g(y_0) = \pi^{-1} \int_{|x_1 - y_1^0| < \delta} \frac{y_2(g(x_1) - g(y_0^1))dx_1}{(x_1 - y_1)^2 + y_2^2} +$$

$$+\pi^{-1} \int_{|x_1 - y_1^0| \geq \delta} \frac{y_2(g(x_1) - g(y_0^1))dx_1}{(x_1 - y_1)^2 + y_2^2} = I_1 + I_2. \tag{15}$$

In view of (10) and (12), it follows that $|I_1| \le \epsilon$. Moreover, by the same change of variable as above, we have:

$$\int_{-\infty}^{y_1^0-\delta} \frac{y_2 dx_1}{(x_1-y_1)^2+y_2^2} = \int_{-\infty}^{\frac{y_1^0-\delta-y_1}{y_2}} \frac{dz}{1+z^2} \to 0, \text{ as } y_1 \to y_0^1,\ y_2 \downarrow 0.$$

Similarly,

$$\int_{y_1^0+\delta}^{\infty} \frac{y_2 dx_1}{(x_1-y_1)^2+y_2^2} = \int_{\frac{y_1^0+\delta-y_1}{y_2}}^{\infty} \frac{dz}{1+z^2} \to 0, \text{as } y_1 \to y_0^1,\ y_2 \downarrow 0.$$

It follows that $\lim_{y\to y_0} I_2 = 0$, as g is bounded on $\mathbb{R}$. This in conjunction with $|I_1| \le \epsilon$ implies $\limsup_{y\to y_0} |u(y) - g(y_0) \le \epsilon, \forall \epsilon > 0$, so (13) is valid.

Under the hypotheses on g_1, the function u_1 is well defined and of class $C^1(\bar{\Omega})$. Differentiating under the integral sign we obviously obtain:

$$-\frac{\partial u_1(y)}{\partial y_2} = \pi^{-1} \int_{-\infty}^{\infty} \frac{y_2 g_1(x_1) dx_1}{(x_1-y_1)^2+y_2^2}.$$

On the basis of the previous discussion, this implies $\dfrac{du_1}{d\eta_y} = -\dfrac{\partial u_1(y)}{\partial y_2} = g_1(y)$ on the boundary $\partial\Omega$ of Ω as g_1 is bounded, which completes the proof.

3.2 Construction of the Green function for the Neumann Problem for $\mathbb{R}^3_+$

We build the Green functions G and G_1 for the (DP) and (NP), respectively, in the situation where $\Omega = \mathbb{R}^3_+$, and then we verify directly that the representation formulas of the functions u and u_1 obtained by means of G and G_1 do indeed provide us with solutions of the (DP) and (NP), respectively.

The following classical result will be needed to achieve that (Theorem 1.1.11, [1]).

Theorem 2 *[1] Let $f : \mathbb{R}^n \to (-\infty, \infty)$ be a measurable function and let $R > 0$.*

i) If $|f(x)| \le c\|x\|^{-\lambda}$, for all $x \in \mathbb{R}^n$ with $\|x\| \le R$, where c is a positive constant, and $\lambda < n$, then $\displaystyle\int_{\|x\|\le R} |f(x)|dx < +\infty$.

ii) If $|f(x)| \le c\|x\|^{-\lambda}$, for all $x \in \mathbb{R}^n$ with $\|x\| \ge R$, where c is a positive constant, and $\lambda > n$, then $\displaystyle\int_{\|x\|\ge R} |f(x)|dx < +\infty$.

Next we construct the Green function for the Neumann problem for the positive half-space $\mathbb{R}^3_+$.

Let $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, x_3 > 0\}$ be the positive half-space. Clearly, $\partial\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, x_3 = 0\} = \{x = (x_1, x_2, 0), (x_1, x_2) \in \mathbb{R}^2\}$. For $y = (y_1, y_2, y_3)$ define by reflection $y^* = (y_1, y_2, -y_3)$. Then the function:

$$G(x,y) = \frac{1}{4\pi\|x-y\|} - \frac{1}{4\pi\|x-y^*\|},\ x \in \bar{\Omega},\ y \in \Omega,\ x \neq y, \tag{16}$$

is a Green function for the (DP) with $f = 0$, and

$$G_1(x,y) = \frac{1}{4\pi\|x-y\|} + \frac{1}{4\pi\|x-y^*\|},\quad x \in \bar{\Omega},\ y \in \Omega,\ x \neq y, \tag{17}$$

is a Green type function for (NP) with $f = 0$.

As $\|x-y\| = \|x-y^*\| = r$ for $y \in \Omega,\ x \in \partial\Omega$, it follows that $G(x,y) = 0,\ x \in \partial\Omega,\ y \in \Omega$.

The outward normal η_x to $\partial\Omega = \{x = (x_1,x_2,x_3) \in \mathbb{R}^3, x_3 = 0\}$ at $x \in \partial\Omega$ is $\eta_x = (0,0,-1)$. The normal derivative of G is $\frac{dG}{d\eta_x} = -\frac{\partial G}{\partial x_3} = \frac{y_3}{2\pi r^3}$ with $x = (x_1,x_2,0), y = (y_1,y_2,y_3), y_3 > 0$, where $r^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2$. Similarly we can check that $\frac{dG_1}{d\eta_x} = 0$, for $x \in \partial\Omega,\ y \in \Omega$.

The formula (3) suggests that a solution to the problem: $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, could be

$$u(y) = \int_{\partial\Omega} u(x)\frac{dG}{d\eta_x}(x,y)\,d\sigma_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y_3 g(x_1,x_2)dx_1dx_2}{[(x_1-y_1)^2+(x_2-y_2)^2+y_3^2]^{\frac{3}{2}}},$$

with $x = (x_1,x_2,0),\ g(x_1,x_2,0) = g(x_1,x_2),\ y = (y_1,y_2,y_3),\ y_3 > 0.$ (18)

Clearly $G_1(x,y) = \frac{1}{2\pi\|x-y\|}$, for $x = (x_1,x_2,0),\ y \in \Omega$. Therefore, (3) suggests that a possible solution to the problem $\Delta u = 0$ in Ω, $\frac{du}{d\eta_x} = g_1$ on $\partial\Omega$, could be

$$u_1(y) = -\int_{\partial\Omega} G_1(x,y)g_1(x)d\eta_x = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g_1(x_1,x_2)}{\|x-y\|}dx_1dx_2, \tag{19}$$

with $x = (x_1,x_2,0), g_1(x_1,x_2,0) = g_1(x_1,x_2), y = (y_1,y_2,y_3),\ y_3 > 0$.

The functions g and g_1 must guarantee the convergence of the improper integrals (18) and (19), respectively (i.e., the existence of u and u_1), and the fact that these functions u and u_1 are solutions of the above Dirichlet and Neumann problems with $f = 0$. An important case in which these requirements are fulfilled is given by:

Theorem 3 *Let g be continuous and bounded on $\mathbb{R}^2$, and g_1 be continuous and with compact support in $\mathbb{R}^2$. Then the functions u and u_1 given by the improper integrals (18) and (19) satisfy:*
1. $u \in C^2(\Omega) \cap C(\bar{\Omega})), \Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$
2. $u_1 \in C^2(\Omega) \cap C^1(\bar{\Omega}),\ \Delta u_1 = 0$ in Ω, $\frac{du_1}{d\eta_x} = g_1$ on $\partial\Omega$.

Proof. We will use the formula

$$I_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y_3 dx_1dx_2}{[(x_1-y_1)^2+(x_2-y_2)^2+y_3^2]^{\frac{3}{2}}} = 1, \tag{20}$$

where $x = (x_1, x_2, 0)$, $y = (y_1, y_2, y_3)$, $y_3 > 0$.

To show (20), we make the change of variables $x_1 - y_1 = y_3 z$, $x_2 - y_2 = y_3 w$, so $dx_1 = y_3 dz$, $dx_2 = y_3 dw$. The integral I_0 becomes

$$I_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dzdw}{(z^2+w^2+1)^{3/2}}.$$

Since $\displaystyle\int \frac{dt}{(t^2+a^2)^{3/2}} = \frac{t}{a^2\sqrt{t^2+a^2}} + C$, we get

$$I_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{t_1\to-\infty}\lim_{t_2\to\infty}\int_{t_1}^{t_2}\frac{dz}{(z^2+w^2+1)^{3/2}}\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{t_1\to-\infty}\lim_{t_2\to\infty}\left(\frac{z}{(w^2+1)\sqrt{z^2+w^2+1}}|_{t_1}^{t_2}\right)\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{t_2\to\infty}\frac{t_2}{(w^2+1)\sqrt{t_2^2+w^2+1}} - \lim_{t_1\to-\infty}\frac{t_1}{(w^2+1)\sqrt{t_1^2+w^2+1}}\right]dw$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{w^2+1}dw = \frac{1}{\pi}\lim_{t_1\to-\infty}\lim_{t_2\to\infty}\int_{t_1}^{t_2}\frac{1}{w^2+1}dw$$

$$= \frac{1}{\pi}\lim_{t_1\to-\infty}\lim_{t_2\to\infty}(\arctan t_2 - \arctan t_1) = 1.$$

The fact that u and u_1 are harmonic on Ω follows by differentiating under the integral sign and taking into account that $G_1(x,y) = G_1(y,x)$, $x \in \bar{\Omega}$, $y \in \Omega$, $x \neq y$, $\Delta_y G_1(x,y) = \Delta_y G_1(y,x) = 0$, $y \in \Omega$, $x \in \partial\Omega$, and $\Delta_y \frac{dG}{d\eta_x}(x,y) = \frac{d}{d\eta_x}(\Delta_y G(x,y))$, $G(x,y) = G(y,x)$ for $y \neq x$, $x \in \bar{\Omega}$, $y \in \Omega$, $\Delta_y G(x,y) = \Delta_y G(y,x) = 0$ for $y \in \Omega$, $x \in \partial\Omega$.

As g is bounded on $\mathbb{R}^2$, and g_1 is continuous and with compact support in $\mathbb{R}^2$, u and u_1 given by (18) and (19) are bounded due to Theorem 2.

Let us to prove that $u(x) = g(x)$ for $x = (x_1, x_2, 0)$, i.e., that

$$\lim_{y\to\bar{y}} u(y) = g(\bar{y}), \tag{21}$$

for all $\bar{y} = (\bar{y}_1, \bar{y}_2, 0)$, $y = (y_1, y_2, y_3)$, with $y_3 > 0$.

In view of (18) and (20), we can write

$$u(y) - g(\bar{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y_3(g(x_1,x_2) - g(\bar{y}_1,\bar{y}_2))dx_1dx_2}{[(x_1-y_1)^2+(x_2-y_2)^2+y_3^2]^{\frac{3}{2}}}, \tag{22}$$

where $y = (y_1, y_2, y_3)$, $y_3 > 0$, $\bar{y} = (\bar{y}_1, \bar{y}_2, 0)$, $x = (x_1, x_2, 0)$, $(x_1, x_2) \in \mathbb{R}^2$.

Given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$|g(x_1,x_2) - g(\bar{y}_1,\bar{y}_2)| < \epsilon, \text{ for } \|(x_1,x_2) - (\bar{y}_1,\bar{y}_2)\| < \delta.$$

We obtain

$$u(y) - g(\bar{y}) = \frac{1}{2\pi} \int\int_{\|(x_1,x_2)-(\bar{y}_1,\bar{y}_2)\|<\delta} \frac{y_3(g(x_1,x_2) - g(\bar{y}_1,\bar{y}_2))dx_1dx_2}{[(x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2]^{\frac{3}{2}}}$$

$$+\frac{1}{2\pi} \int\int_{\|(x_1,x_2)-(\bar{y}_1,\bar{y}_2)\|\geq\delta} \frac{y_3(g(x_1,x_2) - g(\bar{y}_1,\bar{y}_2))dx_1dx_2}{[(x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2]^{\frac{3}{2}}} = I_1 + I_2. \qquad (23)$$

From (20), it follows that $|I_1| \leq \epsilon$.

We will show that the integral I_2 converges to zero for $y \to \bar{y}$.

If $\|y - \bar{y}\| < \frac{\delta}{2}$, $\|x - \bar{y}\| \geq \delta$, we get

$$\|x - \bar{y}\| \leq \|x - y\| + \|y - \bar{y}\| \leq \|x - y\| + \frac{\delta}{2} \leq \|x - y\| + \frac{1}{2}\|x - \bar{y}\|,$$

and so $\|x - y\| \geq \frac{1}{2}\|x - \bar{y}\|$.

Since g is bounded, let $M > 0$ be a constant such that $|g(x)| \leq M$, for all $x \in \mathbb{R}^2$.

Then

$$|I_2| \leq \frac{2M}{2\pi} \int\int_{\|(x_1,x_2)-(\bar{y}_1,\bar{y}_2)\|\geq\delta} \frac{y_3dx_1dx_2}{[(x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2]^{\frac{3}{2}}}$$

$$\leq \frac{8My_3}{\pi} \int\int_{\|(x_1,x_2)-(\bar{y}_1,\bar{y}_2)\|\geq\delta} \frac{dx_1dx_2}{[(x_1-\bar{y}_1)^2 + (x_2-\bar{y}_2)^2]^{\frac{3}{2}}} \to 0,$$

as $y \to \bar{y}$, i.e., as $y_3 \to 0+$, because the above integral is bounded in view of Theorem 2.

We conclude that $\lim_{y\to\bar{y}} I_2 = 0$, which in conjunction with $|I_1| \leq \epsilon$ implies $\limsup_{y\to\bar{y}} |u(y) - g(\bar{y})| \leq \epsilon$, for all $\epsilon > 0$, so (21) is valid.

Under the hypotheses on g_1 the function u_1 is well defined and of class $C^1(\bar{\Omega})$. Differentiating under the integral sign we obviously obtain:

$$-\frac{\partial u_1(y)}{\partial y_3} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y_3 g_1(x_1,x_2)dx_1dx_2}{[(x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2]^{\frac{3}{2}}}.$$

On the basis of the previous discussion, this implies:

$\frac{du_1}{d\eta_y} = -\frac{\partial u_1(y)}{\partial y_3} = g_1(y)$ on the boundary $\partial\Omega$ of Ω as g_1 is bounded, which completes the proof.

3.3 Construction of the Green function for the Neumann Problem for $\mathbb{R}^n_+$

Next we generalize the result of section 3.2 by constructing the Green function for the Neumann problem for the positive half-space $\mathbb{R}^n_+$, $n > 2$.

Let $\Omega = \mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\}$, $n > 2$, be the positive half-space. Clearly, $\partial\Omega = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_n = 0\} = \{x = (x_1, \ldots, x_{n-1}, 0), (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}$.

For $y = (y_1, y_2, \ldots, y_n)$ define by reflection $y^* = (y_1, \ldots, y_{n-1}, -y_n)$.

Then the function:

$$G(x,y) = -\psi(x,y) + \psi(x,y^\star),\ x \in \bar{\Omega},\ y \in \Omega,\ x \neq y, \text{ i.e.,}$$

$$G(x,y) = \frac{1}{(n-2)\sigma_n \|x-y\|^{n-2}} - \frac{1}{(n-2)\sigma_n \|x-y^*\|^{n-2}}, \tag{24}$$

is a Green function for the (DP) with $f = 0$, and

$$G_1(x,y) = -\psi(x,y) - \psi(x,y^\star),\ x \in \bar{\Omega},\ y \in \Omega,\ x \neq y, \text{ i.e.,}$$

$$G_1(x,y) = \frac{1}{(n-2)\sigma_n \|x-y\|^{n-2}} + \frac{1}{(n-2)\sigma_n \|x-y^*\|^{n-2}}, \tag{25}$$

is a Green type function for (NP) with $f = 0$.

As $\|x-y\| = \|x-y^*\| = r$ for $y \in \Omega,\ x \in \partial\Omega$, it follows that $G(x,y) = 0$, $x \in \partial\Omega$, $y \in \Omega$.

The outward normal η_x to $\partial\Omega$ at $x \in \partial\Omega$ is $\eta_x = (0,\ldots,0,-1)$. The normal derivative of G is $\dfrac{dG}{d\eta_x} = -\dfrac{\partial G}{\partial x_n} = \dfrac{2y_n}{\sigma_n r^n}$, $x = (x_1,\ldots,x_{n-1},0)$, $y = (y_1,\ldots,y_{n-1},y_n)$, $y_n > 0$, where $r^2 = (x_1-y_1)^2 + \ldots + (x_{n-1}-y_{n-1})^2 + y_n^2$. Similarly we can check that $\dfrac{dG_1}{d\eta_x} = 0$, for $x \in \partial\Omega$, $y \in \Omega$.

The formula (3) suggests that a solution to the problem: $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, could be:

$$u(y) = \int_{\partial\Omega} u(x)\frac{dG}{d\eta_x}(x,y)\,d\sigma_x = \frac{2y_n}{\sigma_n}\int_{\partial\mathbb{R}^n_+} \frac{g(x)}{\|x-y\|^n} d\sigma_x$$

$$= \frac{2y_n}{\sigma_n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{g(x_1,\ldots,x_{n-1})dx_1\ldots dx_{n-1}}{[(x_1-y_1)^2+\ldots+(x_{n-1}-y_{n-1})^2+y_n^2]^{\frac{n}{2}}}, \tag{26}$$

$x = (x_1,\ldots,x_{n-1},0)$, $g(x_1,\ldots,x_{n-1},0) = g(x_1,\ldots,x_{n-1})$, $y = (y_1,\ldots,y_n)$, $y_n > 0$.

The function u is the solution of the (DP) with $f = 0$ (see [4] for the proof of the Poisson's formula for the half-space of $\mathbb{R}^n_+$, $n > 2$).

Clearly $G_1(x,y) = \dfrac{2}{(n-2)\sigma_n\|x-y\|^{n-2}}$, for $x = (x_1,\ldots,x_{n-1},0)$, $y \in \mathbb{R}^n_+$.

Therefore, (3) suggests that a possible solution to the problem $\Delta u = 0$ in Ω, $\dfrac{du}{d\eta_x} = g_1$ on $\partial\Omega$, could be

$$u_1(y) = -\int_{\partial\mathbb{R}_n^+} G_1(x,y)g_1(x)d\sigma_x = -\frac{2}{(n-2)\sigma_n}\int_{\partial\mathbb{R}^n_+}\frac{g_1(x)}{\|x-y\|^{n-2}}d\sigma_x, \tag{27}$$

$x = (x_1,\ldots,x_{n-1},0), g_1(x_1,\ldots,x_{n-1},0) = g_1(x_1,\ldots,x_{n-1}), y = (y_1,\ldots,y_n)$, $y_n > 0$.

The function g_1 must guarantee the convergence of the improper integral (27) (i.e., the existence of u_1), and the fact that this function u_1 is solution of the Neumann problems with $f = 0$. An important case in which these requirements are fulfilled is given by:

Theorem 4 *Let g_1 be continuous and with compact support in $\mathbb{R}^{n-1}$.*
Then the function u_1 given by the improper integral (27) satisfies:

$$u_1 \in C^2(\Omega) \cap C^1(\bar{\Omega}),\ \Delta u_1 = 0 \ \textit{in} \ \ \Omega, \quad \frac{du_1}{d\eta_x} = g_1 \quad \textit{on} \ \ \partial\Omega.$$

Proof. We will use the formula

$$I_0 = \frac{2y_n}{\sigma_n} \int_{\partial\mathbb{R}^n_+} \frac{1}{\|x-y\|^n} d\sigma_x = 1, \tag{28}$$

where $x = (x_1, \ldots, x_{n-1}, 0)$, $y = (y_1, \ldots, y_n)$, $y_n > 0$.

The fact that u_1 are harmonic on Ω follows by differentiating under the integral sign and taking into account that $G_1(x,y) = G_1(y,x)$, $x \in \bar{\Omega}$, $y \in \Omega$, $x \neq y$, $\Delta_y G_1(x,y) = \Delta_y G_1(y,x) = 0$, $y \in \Omega$, $x \in \partial\Omega$.

As g_1 is continuous and with compact support in $\mathbb{R}^{n-1}$, u_1 defined by (27) is bounded due to Theorem 2.

Let us to prove that $\dfrac{du}{d\eta_x} = g_1(x)$ for $x = (x_1, \ldots, x_{n-1}, 0)$, i.e., that

$$\lim_{y \to \bar{y}} \frac{du}{d\eta_y} = g_1(\bar{y}), \tag{29}$$

for all $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_{n-1}, 0)$ fixed, and $y = (y_1, \ldots, y_n)$, $y_n > 0$.

Under the hypotheses on g_1 the function u_1 is well defined and of class $C^1(\bar{\Omega})$. Differentiating under the integral sign we obviously obtain:

$$\frac{du_1}{d\eta_y} = -\frac{\partial u_1(y)}{\partial y_n} = \frac{2}{\sigma_n} \int_{\partial\mathbb{R}^n_+} \frac{y_n g_1(x)}{\|x-y\|^n} d\sigma_x.$$

In view of (27) and (28), we can write

$$\frac{du_1}{d\eta_y} - g(\bar{y}) = \frac{2y_n}{\sigma_n} \int_{\partial\mathbb{R}^n_+} \frac{g_1(x) - g_1(\bar{y})}{\|x-y\|^n} d\sigma_x,$$

where $y = (y_1, \ldots, y_n)$, $y_n > 0$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_{n-1}, 0)$, $x = (x_1, \ldots, x_{n-1}, 0)$, $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$|g_1(x) - g_1(\bar{y})| < \epsilon, \text{ for } \|x - \bar{y}\| < \delta.$$

Then if $\|y - \bar{y}\| < \dfrac{\delta}{2}$, $y \in \mathbb{R}^n_+$, we have

$$\frac{du_1}{d\eta_y} - g(\bar{y}) = \frac{2y_n}{\sigma_n} \int_{\partial\mathbb{R}^n_+ \cap B(\bar{y},\delta)} \frac{g_1(x) - g_1(\bar{y})}{\|x-y\|^n} d\sigma_x + \frac{2y_n}{\sigma_n} \int_{\partial\mathbb{R}^n_+ \setminus B(\bar{y},\delta)} \frac{g_1(x) - g_1(\bar{y})}{\|x-y\|^n} d\sigma_x$$

$$= I_1 + I_2, \tag{30}$$

where $B(\bar{y},\delta)=\{x\in\mathbb{R}^n;\ \|x-\bar{y}\|<\delta\}$.

From (28), it follows that $|I_1|\le\epsilon$.

If $\|y-\bar{y}\|<\frac{\delta}{2}$, $\|x-\bar{y}\|\ge\delta$, we get

$$\|x-\bar{y}\|\le\|x-y\|+\|y-\bar{y}\|\le\|x-y\|+\frac{\delta}{2}\le\|x-y\|+\frac{1}{2}\|x-\bar{y}\|,$$

and so $\|x-y\|\ge\frac{1}{2}\|x-\bar{y}\|$.

Since g_1 is bounded, let $M>0$ be a constant such that $|g_1(x)|\le M$, for all $x\in\mathbb{R}^{n-1}$. Then

$$\begin{aligned}|I_2| &\le 2M\int_{\partial\mathbb{R}^n_+\setminus B(\bar{y},\delta)}\frac{2y_n}{\sigma_n\|x-y\|^n}d\sigma_x\\ &\le\frac{2^{n+2}My_n}{\sigma_n}\int_{\partial\mathbb{R}^n_+\setminus B(\bar{y},\delta)}\frac{1}{\|x-\bar{y}\|^n}d\sigma_x\to 0,\end{aligned}$$

as $y\to\bar{y}$, i.e., as $y_n\to 0+$, because the above integral is bounded in view of Theorem 2.

We conclude that $\lim_{y\to\bar{y}} I_2=0$, which in conjunction with $|I_1|\le\epsilon$ implies $\limsup_{y\to\bar{y}}|\frac{du}{d\eta_y}-g_1(\bar{y})|\le\epsilon$, for all $\epsilon>0$, so (29) is valid, which completes the proof.

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