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The Dirichlet Problem for a Plate on an Elastic Foundation

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1. Formulation of the Problem

Suppose that a thin elastic plate occupies a bounded domain $S \subset \mathbb{R}^2$ with a simple, closed, and sufficiently smooth (Lyapunov) contour ∂S. The plate experiences plane deformation, which is described by the displacement vector-valued function $u(x) = (u_1(x), u_2(x))^T$, where $x = (x_1, x_2)$ and the superscript T denotes matrix transposition. We assume that the plate lies on an elastic foundation and that the interaction is described by a matrix $K = \operatorname{diag}\{k, k\}$, where $k > 0$ is the corresponding elastic coefficient. The boundary value problem of equilibrium of the plate subject to transverse external forces $q(x) = (q_1(x), q_2(x))^T$ with the Dirichlet boundary condition has the form

$$\begin{aligned} Zu = Au - Ku = q \quad &\text{in } S,\\ u = f \quad &\text{on } \partial S, \end{aligned}$$

where

$$A = \begin{pmatrix} \mu\Delta + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 \\ (\lambda+\mu)\partial_1\partial_2 & \mu\Delta + (\lambda+\mu)\partial_2^2 \end{pmatrix}, \quad \partial_i = \frac{\partial}{\partial x_i}, \ \ i = 1, 2.$$

Let $Hm(S)$, $\mathring{H}_m(S)$, and $H_m(\partial S)$, $m \in \mathbb{R}$, be the standard Sobolev spaces on S and ∂S, respectively. If $q \in \mathring{H}_{-1}(S)$ and $f \in H_{1/2}(\partial S)$, then the unique solvability of the Dirichlet problem in S can easily be proved by standard variational methods [1]. In what follows we consider only the case $q = 0$ because the general case may be reduced to this one. The aim of this article is to present two versions (indirect and direct) of boundary integral equations for the above problem. The indirect one follows from the representation of the solution as a double-layer potential, whereas the direct one is obtained via a limiting process in the Somigliana formula. Finally, we solve the ensuing boundary integral equations approximately and compare the numerical results.

Remark 1 All the statements in this article are valid for much wider classes of domains.

Thus, it suffices to assume that ∂S is a piecewise smooth curve of class $C^{0,1}$ (Lipschitz curve) which consists of a finite number of Lyapunov arcs.

Remark 2 A detailed description of the results on the unique solvability of the systems of boundary integral equations in the allied equilibrium and oscillation problems of bending of thin elastic plates with transverse shear deformation can be found in [2]–[4].

2. The Single-Layer and Double-Layer Potentials

Let $D(x,y)$ be a matrix of fundamental solutions for the operator Z. A simple calculation shows that

$$D(x,y) = (\operatorname{adj} Z)(\partial_x)[t(x,y)I_2],$$

where $\operatorname{adj} Z(\partial_x)$ is the matrix differential operator adjoint to Z acting on $t(x,y)$ with respect to x, $I_2 = \operatorname{diag}\{1,1\}$,

$$t(x,y) = -2[2\pi k(\lambda+\mu)]^{-1}[K_0(c_1|x-y|) - K_0(c_2|x-y|)],$$

$c_1^2 = k/\mu$, $c_2^2 = k/(\lambda+2\mu)$, and K_0 is the modified Bessel function of the second kind and order zero [2]. We also introduce the matrix of singular solutions $P(x,y) = [T(\partial_y)D(y,x)]^T$, where $T(\partial_y)$ is the boundary stress operator

$$T = \begin{pmatrix} (\lambda+2\mu)\nu_1\partial_1 + \mu\nu_2\partial_2 & \mu\nu_2\partial_1 + \lambda\nu_1\partial_2 \\ \lambda\nu_2\partial_1 + \mu\nu_1\partial_2 & (\lambda+2\mu)\nu_2\partial_2 + \mu\nu_1\partial_1 \end{pmatrix}$$

acting on $D(x,y)$ with respect to y and $\nu(x) = (\nu_1(x), \nu_2(x))^T$ is the unit outward normal to ∂S. For each $x \in \mathbb{R}^2$, we introduce the single-layer and double-layer potentials with densities φ and ψ defined on ∂S by

$$(V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)\,ds(y),$$

$$(W\psi)(x) = \int_{\partial S} P(x,y)\psi(y)\,ds(y),$$

respectively. If $\varphi \in H_{-1/2}(\partial S)$ and $\psi \in H_{1/2}(\partial S)$, then $V\varphi$ and $W\psi$ are elements of $H_1(S)$. Denoting by γ the trace operator that maps $H_1(S)$ continuously onto $H_{1/2}(\partial S)$, we introduce boundary operators V^+ and W^+ by

$$V^+\varphi = \gamma(V\varphi), \quad W^+\psi = \gamma(W\psi).$$

It is very easy to verify that V^+ is an integral operator with a weakly singular kernel and $W^+ = -\frac{1}{2}I + W_0$, where I is the identity operator and W_0 is an integral operator with a singular kernel, which means that the corresponding integral is understood in the sense of principal value.

Theorem 1 *The operators V^+ and W^+ define bijections*

$$V^+ : H_{-1/2}(\partial S) \to H_{1/2}(\partial S), \quad W^+ : H_{1/2}(\partial S) \to H_{1/2}(\partial S).$$

3. Boundary Integral Equations

We begin with the indirect method in the Dirichlet problem and represent the solution in the form

$$u(x) = (W\psi)(x), \quad x \in S,$$

with an unknown density ψ. After the limiting transition of the point x to ∂S, we arrive at the system of singular integral equations

$$W^+\psi = f,$$

or, what is the same,

$$-\tfrac{1}{2}\,\psi + W_0\psi = f. \tag{1}$$

Theorem 2 *For every prescribed $f \in H_{1/2}(\partial S)$, system (1) has a unique solution $\psi \in H_{1/2}(\partial S)$. In this case, $u = W\psi \in H_1(S)$ is the weak (variational) solution of the Dirichlet problem.*

We remark that the function ψ has no direct mechanical meaning.

We now turn to the direct version of the boundary integral equations in the same problem. Proceeding just as in the derivation of the third Green formula in the harmonic case, we obtain its analogue, called the Somigliana formula. For functions $u \in C^2(S) \cap C^1(\bar{S})$ such that $Zu = 0$ in S, this formula is

$$\int\limits_{\partial S} [D(x,y)(Tu)(y) - P(x,y)u(y)]\,ds(y) = \begin{cases} u(x), & x \in S, \\ \tfrac{1}{2}\,u(x), & x \in \partial S. \end{cases} \tag{2}$$

Since in the Dirichlet problem we have $\gamma u = f$, from (2) it follows that $V^+\varphi - W^+f = f$, or

$$V^+\varphi = \left(\tfrac{1}{2}\,I + W_0\right)f, \tag{3}$$

where $\varphi = Tu$. We point out that the density φ in (3) has direct mechanical significance, being the boundary moment–stress vector for the plate, computed from the solution of the Dirichlet problem. By Theorem 1, the operators occurring in (3) can be extended by continuity to the corresponding Sobolev spaces.

Theorem 3 *For every prescribed $f \in H_{1/2}(\partial S)$, system (3) has a unique solution $\varphi \in H_{-1/2}(\partial S)$. In this case, $u = V\varphi \in H_1(S)$ is the weak (variational) solution of the Dirichlet problem.*

4. Numerical Example

Consider a square steel floor that occupies the region $\bar{S} = [0,1] \times [0,1]$. Its Lamé coefficients are $\lambda = 1.141 \times 10^8$, $\mu = 8.262 \times 10^7$, and the elastic coefficient of the foundation is assumed to be $k = 4 \times 10^7$. All units are given in SI (kg, m, sec). The boundary condition $f = \gamma u$ is

$$u(x_1, 0) = (0, 0.01\sin(\pi x_1))^T, \qquad u(x_1, 1) = (0, -0.01\sin(\pi x_1))^T,$$
$$u(0, x_2) = (-0.01\sin(\pi x_2), 0)^T, \quad u(1, x_2) = (0.01\sin(\pi x_2), 0)^T.$$

Calculations were performed with the package MATHEMATICA, matrices $D(x, y)$ and $P(x, y)$ were computed symbolically, the approximate solutions of the boundary integral equations were obtained by means of cubic splines, and the Cauchy principal values were computed with Gaussian quadrature. In the table below we present the values of u_i^{ind} and u_i^{dir}, $i = 1, 2$, in the indirect and direct methods, respectively, at four points x in the plate.

$x = (x_1, x_2)$	(0.125,0.375)	(0.400,0.800)	(0.750,0.125)	(0.625,0.750)
u_1^{ind}	−0.00740122	−0.00147556	0.00245106	0.00213585
u_1^{dir}	−0.00740495	−0.00148554	0.00245310	0.00214173
u_2^{ind}	0.00127837	−0.00623869	0.00586544	−0.00515366
u_2^{dir}	0.00129751	−0.00624786	0.00581380	−0.00515675

5. Conclusions

As can be seen from the table, the results obtained from both the direct and indirect methods are very close, coinciding to the 5th decimal place. This means that, computationally speaking, there is very little to choose between the two methods. One has to bear in mind, however, that the direct method is connected to the physics of the problem in the sense that the unknown function in the integral equation represents the moment–stress vector on the boundary of the domain.

The boundary integral equation technique has important applications in problems arising in continuum mechanics and other fields. It is general, elegant, and powerful in the sense that, when distilled into a boundary element procedure, it generates very good approximations of the exact solution, with an exponential rate of convergence, much better than the finite element method. For this reason, it is a useful tool in the hands of practitioners.

References

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